Exercises to "Standard Model of Particle Physics II"

Winter 2020/21

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Sheet 12 - February 10, 2020

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Lecture webpage: `https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html`

Hand-in of solutions: February 17, 2021 - via e-mail, **before 14:00**

Discussion of solutions: February 17, 2021 - on zoom

Problem 24: *Friedmann Equations* [**10 Points**]

Consider a room with a homogeneous density of matter ρ. In this, distances should scale with a scale $R(t)$, i.e.

$$r = R(t) r_0 \qquad (R(0) = 1).$$

a) Use Newton's law of energy conservation for a mass element of mass m to show that

$$k := \frac{8\pi}{3} G \, \rho \, R^2 \, - \, \dot{R}^2$$

is constant.

b) With this definition for k, show that the Friedmann equation

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8 \, \pi \, G}{3} \rho$$

holds. How will the system behave for $k > 0$, $k < 0$ and $k = 0$?

c) Show that, if a 3-sphere with radius r is embedded in a 4-dimensional Euclidean space with the coordinates x_1, x_2, x_3, x_4 and

i) x_4 is choosen as fictitious coordinate (meaning given in terms of the other three coordinates and the radius of the sphere),

ii) new coordinates $\overline{r}, \theta, \phi$ are introduced, such that $x_1 = \overline{r} \, \sin\theta \, \cos\phi$, $x_2 = \overline{r} \, \sin\theta \, \sin\phi$ and $x_3 = \overline{r} \, \cos\theta$,

iii) the radial coordinate is appropriately rescaled,

the spatial line element can be written in the Friedmann-Robertson-Walker form with $k = 1$.

d) From the first law of thermodynamics $dE = dQ + dW$.
For a moving, and thus thermally closed, volume element show that

$$-3 \, \frac{\dot{R}}{R} = \frac{\dot{\rho}}{p + \rho}$$

holds. *Hint:* remember that the volume scales with R^3.

Problem 25: *Solutions of the Friedmann Equations* [**6 Points**]

a) Consider the flat universe ($k = 0$) immediately after the end of inflation and solve the Friedmann equation in the case of a radiation-dominated universe.

b) At later times the universe is dominated by dark matter. Solve the Freidmann equation in this case too, assuming always a flat universe.

c) The estimated energy density of matter in today's universe (including that of cold dark matter and the one of baryonic non relativistic matter) is $\rho_{\mathrm{M}}^0 = 1.88 \times 10^{-29}\, \Omega_{\mathrm{M}}^0\, h^2$ g cm^{-3}, with $\Omega_{\mathrm{M}}^0 \simeq 0.3$. If we consider the neutrinos to be massless, the radiation energy density in today's universe including the photon energy density and the neutrinos energy density is around $\rho_{\mathrm{r}}^0 = 6.55 \times 10^{-34}$ g cm^{-3}. With this information and taking the value of the Hubble reduced parameter $h = 0.674$ suggested by the Planck data, calculate the scale factor R_{eq} at the time of equality of matter and radiation.

Problem 26: *Special solutions of the Einstein Equations* [**8 Points**]

a) Show with the help of the modified Friedmann Equations

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8\,\pi\,G}{3}\rho + \frac{\Lambda}{3}, \tag{1}$$

$$-2\frac{\ddot{R}^2}{R} - \frac{\dot{R}^2}{R^2} - \frac{k}{R^2} = 8\,\pi\,G\,p - \Lambda, \tag{2}$$

that the equation

$$\dot{\rho} + 3\,\frac{\dot{R}}{R}(\rho + p) = 0 \tag{3}$$

holds also for $\Lambda \neq 0$.

b) Find the solutions of (3) for which $\dot{\rho} = 0$ holds.

c) Show that for a flat universe ($k = 0$) with matter ($p = 0$, $\rho \neq 0$) and cosmological constant ($\Lambda \neq 0$)

$$R(t) = R_0(\sinh At)^{\frac{2}{3}}$$

is a solution of (1) - (3).