Exercises to "Standard Model of Particle Physics II"

Winter 2016/17

Prof. Dr. Manfred Lindner and Dr. Werner Rodejohann Sheet 11 18.01.17

Problem 21: *The Boltzmann equation / WIMP miracle (2)* [**15 Points**]

Consider a stable particle ψ. In a comoving volume, we know that the number of ψ and $\bar{\psi}$ changes only through annihilation and inverse annihilation processes. Let χ denote all possible final states, then we can write:

$$\psi\bar{\psi} \leftrightarrow \chi\bar{\chi}$$

Under certain simplifying assumptions, the Boltzmann equation which rules the evolution of the ψ-number density n_ψ can, together with an equation for the entropy s, be written as:

$$\frac{dn_\psi}{dt} = -3Hn - \langle\sigma_{\text{ann}}v\rangle(n^2 - n_{eq}^2) \tag{1}$$

$$\frac{ds}{dt} = -3Hs \tag{2}$$

Here t is the time, H is the Hubble parameter, n_{eq} and $\langle\sigma_{\text{ann}}b\rangle$ are the WIMP equilibrium number density and the thermally averaged total annihilation cross-section.

a) It is useful to combine equations (1) and (2) into one differential equation. To this end define $Y \equiv \frac{n_\psi}{s}$ and $x \equiv \frac{m}{T}$. Derive the following equation:

$$\frac{dY}{dx} = \frac{1}{3H}\frac{ds}{dx}\langle\sigma_{\text{ann}}v\rangle(Y^2 - Y_{eq}^2)\,. \tag{3}$$

b) According to the Friedmann equations, the Hubble parameter is determined by the mass-energy density ρ as

$$H^2 = \frac{8\pi}{3M_{\text{Pl}}^2}\rho$$

with the Planck mass $M_{\text{Pl}} = 1.22 \times 10^{19}$ GeV. The energy and entropy densities are related to the photon temperature as

$$\rho = \frac{\pi^2}{30}g_{\text{eff}}(T)T^4$$

and

$$s = \frac{2\pi^2}{45}h_{\text{eff}}(T)T^3,$$

where $h_{\text{eff}}(T)$ and $g_{\text{eff}}(T)$ are the effective number of relativistic degrees of freedom for the entropy and energy densities, respectively. Show that in a Friedmann Universe, equation (3) reads:

$$\frac{dY}{dx} = -\left(\frac{45}{\pi M_{\text{Pl}}^2}\right)^{-1/2}\frac{g_*^{1/2}m}{x^2}\langle\sigma_{\text{ann}}v\rangle(Y^2 - Y_{eq}^2) \tag{4}$$

with

$$g_*^{1/2} = \frac{h_{\text{eff}}}{g_{\text{eff}}^{1/2}}\left(1 + \frac{1}{3}\frac{T}{h_{\text{eff}}}\frac{dh_{\text{eff}}}{dT}\right)$$

Unfortunately, equation (4) can only be solved numerically with the initial condition $Y = Y_{eq}$ at $x \approx 1$ to obtain the present WIMP abundance

$$Y_0 \approx \frac{17 + 3 \cdot \ln(m/\,\mathrm{GeV})}{M_{\mathrm{Pl}} \cdot m \cdot \sigma_0 \cdot g_*^{1/2}(T_0)}\,,$$

where the effective number of relativistic degrees of freedom today is $g_*(T_0) \approx 75.75$. Note that we make use of a crude approximation $\langle \sigma_{\mathrm{ann}} v \rangle \approx \sigma_0$ neglecting terms of $\mathcal{O}\left(v^2\right)$ which may only be used in the non-relativistic limit. The solutions to the Boltzmann equation are shown in figure 1.

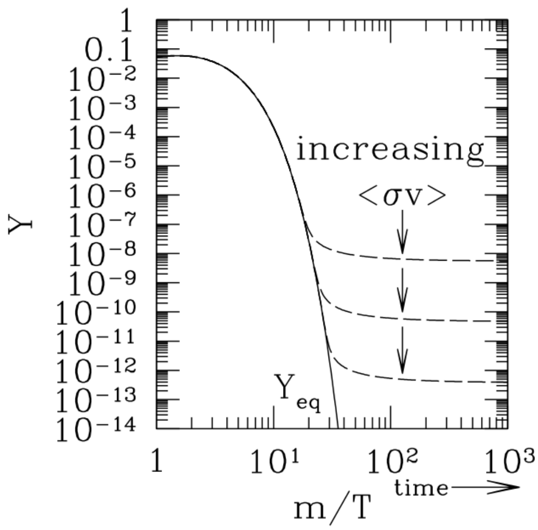

Figure 1: The plot shows the result of equation (4) in terms of $Y(x)$.

c) The entropy density today s_0 can be written in terms of n_γ, which in turn is given by

$$n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3.$$

In addition, $n_{\gamma,0}$ can be related to $n_{b,0}$ via the baryon to photon ratio $\eta \equiv \frac{n_{b,0}}{n_{\gamma,0}} = 6 \cdot 10^{-10}$. Find the value for the dark matter mass m in order to match the observed dark matter to baryon ratio $\frac{\rho_{DM,0}}{\rho_{b,0}} \approx 4 - 6$. Assume that the interaction is of the order of the weak scale, $\sigma_0 \approx G_F^2 \cdot m^2$, with $G_F = 1.15 \cdot 10^{-5}\,\mathrm{GeV}^{-2}$. Is the value you derive for the mass surprising?

Problem 22: *Tremaine-Gunn bound* [**5 Points**]

Assume that neutrinos have a mass, large enough that they are non-relativistic today. This neutrino gas would not be homogeneous, but clustered around galaxies. Assume that they dominate the mass of these galaxies (ignore other matter) as if they were the dark matter. We know the mass $M(r)$ within a given radius r in a galaxy from the velocity $v(r)$ of stars rotating around it [cf. Problem 19 on sheet 10]. The mass could be due to a few species of heavy neutrinos or more species of lighter neutrinos. However, the available phase space limits the number of neutrinos with velocities below the escape velocity from the galaxy (you don't need to assume a thermal distribution). These considerations give a lower limit for the mass of neutrinos m_ν if they dominate the mass of the galaxy. Assume for simplicity that all neutrinos have the same mass, their distribution is spherically symmetric and the escape velocity is independent of the radius r. Furthermore, assume that all states with momenta smaller than the mass times the escape velocity are populated, with the number of states per unit phase space volume in general given by

$$n = g/(2\pi\hbar)^3 \cdot \int_0^\infty f(p)dp \tag{5}$$

where $f(p)$ is the Fermi-Dirac distribution and g describes the relativistic degrees of freedom (assume $g = 2$).

Find a rough estimate for the minimal m_ν if they dominate the mass of a galaxy (are dark matter). Give a numerical value if $v(r) = 220\,\text{km/s}$ at $r = 10\,\text{kpc}$.

From other cosmological bounds we can roughly estimate $m_\nu < 0.3\,\text{eV}$. Compare this value to your value for m_ν, what do you conclude? Can you derive a similar bound for bosons?

Tutor:
Moritz Platscher
e-mail: moritz.platscher@mpi-hd.mpg.de

Lecture webpage: `www.mpi-hd.mpg.de/manitop/StandardModel2/index.html`

Hand-in and discussion of sheet:
Wednesday, 14:15, Phil12, R106