Exercises to "Standard Model of Particle Physics II"

Winter 2020/21

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Lecture webpage: `https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html`

Hand-in of solutions:
February 3, 2021 - via e-mail, **before 14:00**

Discussion of solutions:
February 3, 2021 - on zoom

Problem 20: *Fermion mass matrix diagonalization* [**10 Points**]

a) A hermitian $n \times n$ matrix $M = M^\dagger$ can always be diagonalized by a unitary transformation

$$UMU^\dagger = D = \mathrm{diag}\,(m_1, m_2, \ldots, m_n)$$

where the eigenvalues m_i can be negative as well as positive. Show that one can choose an appropriate bi-unitary transformation $UMV^\dagger$ to diagonalize M so that all diagonal elements are non-negative.

b) Consider an arbitrary complex $n \times n$ matrix A. Show that one can always find two unitary transformation matrices U and V such that $UAV^\dagger$ is diagonal with non-negative elements. Although it is not a formal requirement, you may assume that the eigenvalues of $AA^\dagger$ are non-zero, which will simplify the proof.

c) Now we know that the mass matrices for the SM fermions can be diagonalized by bi-unitary transformations with non-negative mass eigenvalues. The diagonalization in the charged lepton sector leads to a mixing for the charged leptons

$$e_L = U^\dagger e'_L\,, \qquad e_R = V^\dagger e'_R\,.$$

where the primed (unprimed) fields denote the fields in the flavour (mass) basis. Note that since the interactions with W and Z are diagonal in flavor basis, they would mix different mass eigenstates. Show that in the case of massless neutrinos this mixing for the charged leptons would have no physical effect.

d) In the case of quarks, both the up-type and the down-type quarks have a mass matrix, such that diagonalization leads to

$$u_L = U_u^\dagger u'_L\,, \qquad u_R = V_u^\dagger u'_R\,,$$
$$d_L = U_d^\dagger d'_L\,, \qquad d_R = V_d^\dagger d'_R\,.$$

Show that in this case the mixing of quarks *does* leave a physical effect in charged-current (coupling to W interactions). Give, in terms of the U and V matrices above, the 3-by-3 matrix which describes the coupling between u, c, and t quarks in their mass basis with respect to d, s and b quarks in their mass basis in charged-current interactions. This is the famous Cabibbo-Kobayashi-Maskawa (CKM) matrix.

Problem 21: *Seesaw I* [**10 Points**]

The Higgs mechanism generates Dirac masses for the active neutrinos when right-handed sterile neutrinos are introduced. The following term appears in the Lagrange density:

$$\mathscr{L}_{\text{Dirac}} = -\overline{\nu}_{\text{L}} M_{\text{D}} N_{\text{R}} + \text{h.c.}\,,$$

where $\nu_{\text{L}} = (\nu_{\text{L}}^1,\ \nu_{\text{L}}^2,\ \nu_{\text{L}}^3)^{\text{T}}$ is the column vector of the active neutrinos and N_{R} the corresponding vector for the sterile neutrinos. The matrix M_{D} is − in general − a complex 3×3 matrix.
The right-handed sterile neutrinos can, furthermore, have a Majorana mass with the Lagrange density

$$\mathscr{L}_{\text{Majorana}} = -\frac{1}{2}\overline{(N_{\text{R}})^{\text{c}}} M_{\text{R}} N_{\text{R}} + \text{h.c.}\,,$$

where M_{R} is a symmetric 3×3 matrix, and $\psi^{\text{c}} = \mathcal{C}\overline{\psi}^{T}$ for a general Dirac spinor ψ and the charge conjugation matrix $\mathcal{C} = i\gamma^2\gamma^0$.
Let m_{D} and m_{R} denote the mass scale of M_{D} and M_{R}, respectively. Suppose the entries of M_{R} are much larger than the ones of M_{D} ($m_{\text{R}} \gg m_{\text{D}}$).

a) Show that it is possible to rewrite the whole mass matrix in the flavour basis in the following way

$$\mathscr{L}_{\text{mass}} \equiv \mathscr{L}_{\text{Dirac}} + \mathscr{L}_{\text{Majorana}} = -\frac{1}{2}\overline{\Psi^{\text{c}}} M \Psi + \text{h.c.}\,,$$

with

$$\Psi \equiv \begin{pmatrix} (\nu_{\text{L}})^{\text{c}} \\ N_{\text{R}} \end{pmatrix} \qquad \text{and} \qquad M \equiv \begin{pmatrix} 0 & M_{\text{D}} \\ M_{\text{D}}^{\text{T}} & M_{\text{R}} \end{pmatrix}.$$

For the solution prove and use the identity $\overline{\nu_{\text{L}}} M_{\text{D}} N_{\text{R}} = \overline{(N_{\text{R}})^{\text{c}}} M_{\text{D}}^{\text{T}} (\nu_{\text{L}})^{\text{c}}$.

b) Using the (unitary) transformation $\Psi = U\chi$ with $U = \begin{pmatrix} 1 & \rho \\ -\rho^{\dagger} & 1 \end{pmatrix}$ (change of basis) it is possible to convert the 6×6 matrix M into a block diagonal form, i.e. that it takes on the following form

$$U^{\text{T}} M U \simeq \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \tag{1}$$

with symmetric 3×3 matrices $M_1,\ M_2$. The matrix ρ in the transformation matrix U is assumed to be proportional to the scale m_{R}^{-1} and terms of order m_{R}^{-2} (and smaller) can be neglected in the calculation.

Determine ρ, and M_1 and M_2 from Eq. (1). What is the connection between the fields $\chi_1,\ \chi_2$ and the original fields $\nu_{\text{L}},\ N_{\text{R}}$? (You may assume that M_{R} is invertible.)

c) Where does the name “seesaw” come from?