Exercises to "Standard Model of Particle Physics II"

Winter 2018/19

Prof. Dr. Manfred Lindner and Dr. Werner Rodejohann Sheet 10 December 19, 2018

Problem 20: *Neutrino oscillations in matter* [**20 Points**]

The MSW effect describes oscillations of neutrinos in matter. Consider the case of two flavour oscillations in a realistic (non-constant) mass distribution.
The time evolution of the eigenfunctions of the Hamiltonian in matter is described by the following expression:

$$i\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix}|\nu_{\mathrm{A}}\rangle\\ |\nu_{\mathrm{B}}\rangle\end{pmatrix}=\begin{pmatrix}E_{\mathrm{A}}(t) & -i\dot{\theta}(t)\\ i\dot{\theta}(t) & E_{\mathrm{B}}(t)\end{pmatrix}\begin{pmatrix}|\nu_{\mathrm{A}}\rangle\\ |\nu_{\mathrm{B}}\rangle\end{pmatrix},$$

where

$$-E_{\mathrm{A}}(t)=E_{\mathrm{B}}(t)=\Delta m^2\sin(2\theta_0)/4E\sin(2\theta)$$

with the (constant) vacuum mixing angle θ_0 and electron number density $N_e(t)$ in matter. The eigenstates in matter are $|\nu\rangle=(|\nu_{\mathrm{A}}\rangle,\ |\nu_{\mathrm{B}}\rangle)^{\mathrm{T}}$ and at any given time they are connected to the flavor states via: $|\tilde{\nu}\rangle=\tilde{U}(t)|\nu\rangle$ with

$$|\tilde{\nu}\rangle=\begin{pmatrix}|\tilde{\nu}_e\rangle\\ |\tilde{\nu}_\mu\rangle\end{pmatrix},\qquad \tilde{U}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix},$$

and the effective (time-dependent) mixing angle

$$\tan(2\theta)=\frac{\frac{\Delta m^2}{2E}\sin(2\theta_0)}{\frac{\Delta m^2}{2E}\cos(2\theta_0)-\sqrt{2}G_{\mathrm{F}}N_e(t)}\,. \tag{1}$$

The effective Hamiltonian is not diagonal, since matter eigenstates and thus the mixing angle θ are not constant. If, however, $\dot{\theta}$ is small, i.e. the change in θ occurs slowly, we obtain the adiabatic approximation, where we can neglect $\dot{\theta}$. Suppose that at the time $t=t_i$ an electron neutrino is produced,

$$|\tilde{\nu}_\alpha(t_i)\rangle\equiv\delta_{\alpha e}|\tilde{\nu}_\alpha\rangle$$

a) What is the expression for $|\tilde{\nu}_\alpha(t_f)\rangle$ at some later time t_f in the adiabatic approximation?

b) Show that in the adiabatic approximation the result for the transition probability is given by

$$P(\nu_e\to\nu_\mu)=\tfrac{1}{2}-\tfrac{1}{2}\cos(2\theta_i)\cos(2\theta_f)-\tfrac{1}{2}\sin(2\theta_i)\sin(2\theta_f)\cos\Phi_{\mathrm{AB}}\,, \tag{2}$$

with $\theta_i\equiv\theta(t=t_i)$, $\theta_f\equiv\theta(t=t_f)$, and

$$\Phi_{\mathrm{IJ}}\equiv\int_{t_i}^{t_f}\left[E_{\mathrm{I}}(t)-E_{\mathrm{J}}(t)\right]\mathrm{dt}\,,$$

where $\mathrm{I},\ \mathrm{J}\in\{\mathrm{A},\ \mathrm{B}\}$.

c) Show that in the case of constant electron density, Eq. (2) reduces to the standard formula of two-flavor oscillations

$$P(\nu_e\to\nu_\mu)(t)=\sin^2(2\theta)\sin^2\left(\frac{\pi t}{L_M^{\mathrm{osc}}}\right),$$

and determine L_M^{osc}. What happens in the limit $N_e\to 0$?

d) Use Eq. (1) to find an expression for $\sin^2(2\theta)$. The formula for $\sin^2(2\theta)$ describes a Breit-Wigner distribution. We define the resonance width at half height, Δr, as the width at which the curve of the distribution is larger than one half [$\sin^2(2\theta) > 1/2$]. Find an expression for Δr in terms of θ_0.

e) The *adiabaticity parameter* is defined as

$$\gamma_r \equiv \frac{|E_{\mathrm{A}}|}{|\dot{\theta}|},$$

which becomes large if the adiabatic approximation is good. To interpret this, first show that

$$\dot{\theta} = \frac{1}{2}\frac{\sin^2(2\theta)}{\Delta m^2 \sin(2\theta_0)} 2\sqrt{2} E G_F \dot{N}_e .$$

Now assume that the change in matter density is approximately linear, $\dot{N}_e \approx$ const, and that the density passes exactly through the resonance, i.e. $\Delta N_e = N_e(t_f) - N_e(t_i) \propto \Delta r$ to formulate γ_r in terms of L_M^{osc} and $\Delta t = t_f - t_i$. Discuss the adiabaticity condition $\gamma_r > 3$.

Problem 21: *The Koide relation* [**0 Points**]

The Koide formula reads

$$Q = \frac{m_e + m_\mu + m_\tau}{\left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}\right)^2} .$$

a) Assuming only that the masses are positive, determine the range in which Q may take values.

b) Calculate Q for the current best-fit values according to PDG[1], $m_e = 0.510\,998\,946\,1\,\mathrm{MeV}$, $m_\mu = 105.658\,374\,5\,\mathrm{MeV}$, $m_\tau = 1776.86\,\mathrm{MeV}$.

c) The empirical value of Q lies very close to a rational number which has a special position in the range of possible Q-values. Can you find an explanation for this astonishing apparent coincidence?

Happy Holidays!

Tutor:
Ingolf Bischer
e-mail: bischer@mpi-hd.mpg.de

Lecture webpage: `https://www.mpi-hd.mpg.de/manitop/StandardModel2/index.html`

Hand-in and discussion of sheet:
January 9, 2019, 15:45, Philosophenweg 12, R106

[1] http://pdg.lbl.gov/2018/tables/rpp2018-sum-leptons.pdf