Exercises to "Standard Model of Particle Physics II"

Winter 2014/15

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Exercise 22: Seesaw I [**10 Points**]

The Higgs mechanism generates Dirac masses for the active neutrinos, when right-handed sterile neutrinos are introduced. The following term appears in the Lagrange density:

$$\mathscr{L}_{\rm Dirac} = -\overline{\nu}_{\rm L} M_{\rm D} N_{\rm R} + {\rm h.c.},$$

where $\nu_{\rm L} = (\nu_{\rm L}^1,\ \nu_{\rm L}^2,\ \nu_{\rm L}^3)^{\rm T}$ is the column vector of the active neutrinos and $N_{\rm R}$ the corresponding vector for the sterile neutrinos. The matrix $M_{\rm D}$ is $-$ in general $-$ a complex 3×3 matrix.
The right-handed sterile neutrinos can, furthermore, have a Majorana mass with the Lagrange density:

$$\mathscr{L}_{\rm Majorana} = -\frac{1}{2}\overline{(N_{\rm R})^{\rm c}} M_{\rm R} N_{\rm R} + {\rm h.c.},$$

where $M_{\rm R}$ is a symmetric 3×3 matrix.
Let $m_{\rm D}$ and $m_{\rm R}$ denote the mass scale of $M_{\rm D}$ and $M_{\rm D}$, respectively. Suppose the entries of $M_{\rm R}$ are much larger than the ones of $M_{\rm D}$, "$m_{\rm R} \gg m_{\rm D}$".

a) Show that it is possible to rewrite the whole mass matrix in the flavour basis in the following way

$$\mathscr{L}_{\rm mass} \equiv \mathscr{L}_{\rm Dirac} + \mathscr{L}_{\rm Majorana} = -\frac{1}{2}\overline{\Psi^{\rm c}} M \Psi + {\rm h.c.},$$

with

$$\Psi \equiv \begin{pmatrix} (\nu_{\rm L})^{\rm c} \\ N_{\rm R} \end{pmatrix} \qquad \text{and} \qquad M \equiv \begin{pmatrix} 0 & M_{\rm D} \\ M_{\rm D}^{\rm T} & M_{\rm R} \end{pmatrix}$$

For the solution prove and use the identity $\overline{\nu_{\rm L}} M_{\rm D} N_{\rm R} = \overline{(N_{\rm R})^{\rm c}} M_{\rm D}^{\rm T} (\nu_{\rm L})^{\rm c}$.

b) Using the (unitary) transformation $\Psi = U\chi$ with $U = \begin{pmatrix} 1 & \rho \\ -\rho^\dagger & 1 \end{pmatrix}$ (change of basis) it is possible to convert the 6×6 matrix M into a block diagonal form, i.e. that it takes on the following form

$$U^{\rm T} M U \simeq \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \tag{1}$$

with symmetric 3×3 matrices $M_1,\ M_2$. The matrix ρ in the transformation matrix U is assumed to be proportional to the scale $m_{\rm R}^{-1}$ and terms of order $m_{\rm R}^{-2}$ (and smaller) can be neglected in the calculation.

Determine ρ, and M_1 and M_2 from eq. (1). What is the connection between the fields $\chi_1,\ \chi_2$ and the original fields $\nu_{\rm L},\ N_{\rm R}$? (You may assume that $M_{\rm R}$ is invertible.)

c) Where does the name "seesaw" come from?

Exercise 23: Seesaw II [**10 Points**]

We consider the lepton sector of the Standard Model and expand it by adding a Higgs triplet Δ. The particles considered have the following $\mathrm{SU}(2)_\mathrm{L} \otimes \mathrm{U}(1)_\mathrm{Y}$ transformation properties:

$$L_a \sim (2,\ -1); \qquad l_{a\mathrm{R}} \sim (1,\ -2); \qquad \phi \sim (2,\ 1); \qquad \Delta \sim (3,\ 2),$$

where the fields denote (in order of appearance) the left-handed lepton doublet, the right-handed lepton singlet, the SM Higgs doublet and the (non-SM) Higgs triplet, and a is a flavour index. For the triplet use the representation as 2×2 matrix with the charge eigenstates

$$\Delta = \begin{pmatrix} \Delta^+ & \sqrt{2}\Delta^{++} \\ \sqrt{2}\Delta^0 & -\Delta^+ \end{pmatrix}.$$

The mass terms for the leptons stem from the Yukawa Lagrangian

$$\mathscr{L}_\mathrm{Y} = \sum_{a,b} \Big[-y_{ab}\overline{l_{a\mathrm{R}}}\phi^\dagger L_b + \frac{1}{2}\tilde{y}_{ab}\overline{L_a^\mathrm{c}}\mathrm{i}\tau_2\Delta L_b \Big] + \text{h.c.}$$

a) Convince yourself that $\mathscr{L}_\mathrm{Y}$ is a singlet under $\mathrm{SU}(2)_\mathrm{L} \otimes \mathrm{U}(1)_\mathrm{Y}$.

b) The introduction of the triplet changes the Higgs potential to

$$\begin{aligned} V(\phi,\ \Delta) =& a\phi^\dagger\phi + \frac{b}{2}\mathrm{Tr}[\Delta^\dagger\Delta] + c(\phi^\dagger\phi)^2 + \frac{d}{4}(\mathrm{Tr}[\Delta^\dagger\Delta])^2 + \frac{e-h}{2}\phi^\dagger\phi\mathrm{Tr}[\Delta^\dagger\Delta] \\ &+ \frac{f}{4}\mathrm{Tr}[\Delta^\dagger\Delta^\dagger]\mathrm{Tr}[\Delta\Delta] + h\phi^\dagger\Delta^\dagger\Delta\phi + (t\phi^\dagger\Delta(\mathrm{i}\tau_2\phi^*) + \text{h.c.}). \end{aligned}$$

Use the condition that only neutral components of the Higgs fields can develop non-zero vacuum expectation values (vev's) $\langle\phi^0\rangle = v$ and $\langle\Delta^0\rangle = v_\Delta/\sqrt{2}$ and find $V(\langle\phi\rangle,\ \langle\Delta\rangle)$.

c) Define $t = |t|e^{\mathrm{i}\omega}$, $v_\Delta = |v_\Delta|e^{\mathrm{i}\gamma}$ and minimize the potential with respect to $v,\ |v_\Delta|$, and γ. *Hint*: Start with γ.

d) Show that, under the assumptions $a,\ b \propto v^2$, and $c,\ d,\ e,\ f,\ h \propto 1$, together with $|t| \ll v$, the conditions for the minimum are approximately equivalent to

$$v^2 \approx -\frac{a}{2c} \quad \text{and} \quad |v_\Delta| = \frac{2|t|v^2}{b+(e-h)v^2}.$$

e) What do your findings imply for the masses in the lepton sector?

f) Assume that $\sqrt{b} = m_\Delta$ is very large compared to the electroweak scale v, and $|v_\Delta|$, and that $t^2 \propto b$. Find the relations between the vev's under these conditions. What does it mean for the lepton masses?

g) What is the difference between this scenario (the so-called "type II" seesaw) and the type I seesaw from Ex. 22?

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Tutorials homepage: http://www.mpi-hd.mpg.de/manitop/StandardModel2/exercise.html

Hand-in and discussion of sheet:
during tutorial on Thursday, 29.01.15, 9.15 am, kHs, Philosophenweg 12