can describe this as β-function

$$\frac{d\alpha}{d\log Q^2} = \beta_\alpha = \frac{\alpha^2}{3\pi} \left(\equiv \frac{1}{g}\frac{1}{3\pi}\right); \ \beta_\alpha > 0 \Rightarrow \alpha \nearrow \text{ for } Q^2 \nearrow$$

$$\frac{d\alpha}{d\log Q^2} = \frac{d}{d\log Q^2}\frac{1}{g} = -\frac{1}{g^2}\frac{dg}{d\log Q^2} = \frac{1}{3\pi}\frac{1}{g^2}$$

$$\Rightarrow \frac{dg}{d\log Q^2} = -\frac{1}{3\pi} \Rightarrow g(Q^2) = -\frac{1}{3\pi}\log Q^2 + C$$

C from boundary condition $\leftrightarrow$ reference measurement

$$g(\mu^2) = -\frac{1}{3\pi}\log\mu^2 + C \ ; \text{ insert in } g(Q^2)$$

thus: $$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi}\log Q^2/\mu^2}$$

β-function of $SU(N)$ (massless gauge bosons...)

$$\beta(\alpha_s) = -\left(\frac{11}{3}N - \frac{2N_f}{3}\right)\frac{\alpha_s^2}{\pi \cdot 4}$$

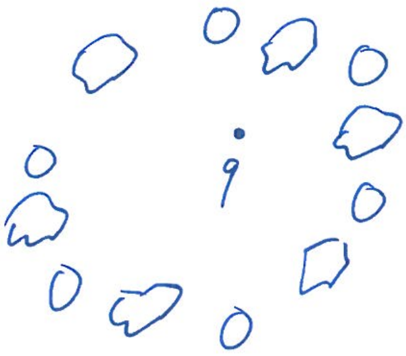
(e.g. QCD: $N_f = 6$, $N = 3$)

$$\Rightarrow \alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{\alpha_s(\mu^2)}{4\pi}\left(\frac{11}{3}N - \frac{2}{3}N_f\right)\log Q^2/\mu^2}$$

for $N=3$: $\alpha_s \searrow$ for $Q^2 \nearrow$ as long as $N_f \leq 16$
"asymptotic freedom"

in analogy: $\alpha_s \nearrow$ for $Q^2 \searrow$ „confinement"

reason:



same effect of fermions as in QED;
opposite effect from gauge-bosons (self interacting)
„anti-screening"

$\alpha_s(M_Z) \simeq 0.12$

$\downarrow$ $\quad \Rightarrow \alpha_s(\Lambda_{QCD}) = \infty \quad \Rightarrow \Lambda \simeq O(100\ \mathrm{MeV})$

$\alpha_s(2m_p) \simeq 0.3$

I 4) The Standard Model Content

(ignore QCD from now on)

$$\boxed{SU(2)_L \times U(1)_Y}$$

$SU(2)_L$ → weak interactions
Fermi interactions

$U(1)_Y$ → Hypercharge

⇒ expect $2^2 - 1 + 1$ gauge bosons

⇒ still need to make 3 of them massive (later)

After choosing symmetry: not done yet!

need to choose representation under group!

$\Psi_1 = \begin{pmatrix} u \\ d \end{pmatrix}_L$ or $\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$ with Hypercharge Y_1, $\tilde{Y}_1$

$\Psi_2 = u_R$, Y_2

$\Psi_3 = d_R$, Y_3 or e_R, $\tilde{Y}_3$

Notation: $\Psi_1 \sim 2_L$ doublets of $SU(2)_L$

$\Psi_{2,3} \sim 1_L$ singlets of $SU(2)_L$

$U(1)_Y$ generated by $\exp\left\{\frac{i}{2} g' \hat{Y} \beta(x)\right\}$

$SU(2)_L$ generated by $\exp\left\{i g \frac{\sigma_i}{2} \alpha_i(x)\right\} \equiv U_L$

$\hat{Y}$ is hypercharge operator : $\hat{Y} u_R = Y_2 u_R$ etc.

Isospin : $\hat{I}_3 \, u_L = \frac{1}{2} u_L$

$\hat{I}_3 \, d_L = -\frac{1}{2} d_L$

$\hat{I}_3 \, u_R = 0$

$\Rightarrow$ Transformation rules :

$$\psi_1 \to \psi_1' = \exp\left\{\frac{i}{2} g' Y_1 \beta(x)\right\} U_L \psi_1 \; ; \; D_\mu = \partial_\mu + i g \frac{\sigma_i}{2} W_\mu^i + \frac{i}{2} g' \hat{Y} B_\mu$$

$$\psi_{\substack{2\\3}} \to \psi_{\substack{2\\3}}' = \exp\left\{\frac{i}{2} g' Y_{\substack{2\\3}} \beta(x)\right\} \psi_{\substack{2\\3}} \quad ; \; D_\mu = \partial_\mu + \frac{i}{2} g' \hat{Y} B_\mu$$

$\Rightarrow$ „chiral theory" treats left-handed and right-handed particles differently.

Need to solve: $\to$ massive gauge bosons

$\to$ QED should be part of SM

II Spontaneous Symmetry Breaking and the Higgs Mechanism

SSB: $\mathcal{L}$ possess symmetry which the ground state does not possess.

II 1) Basic Principle

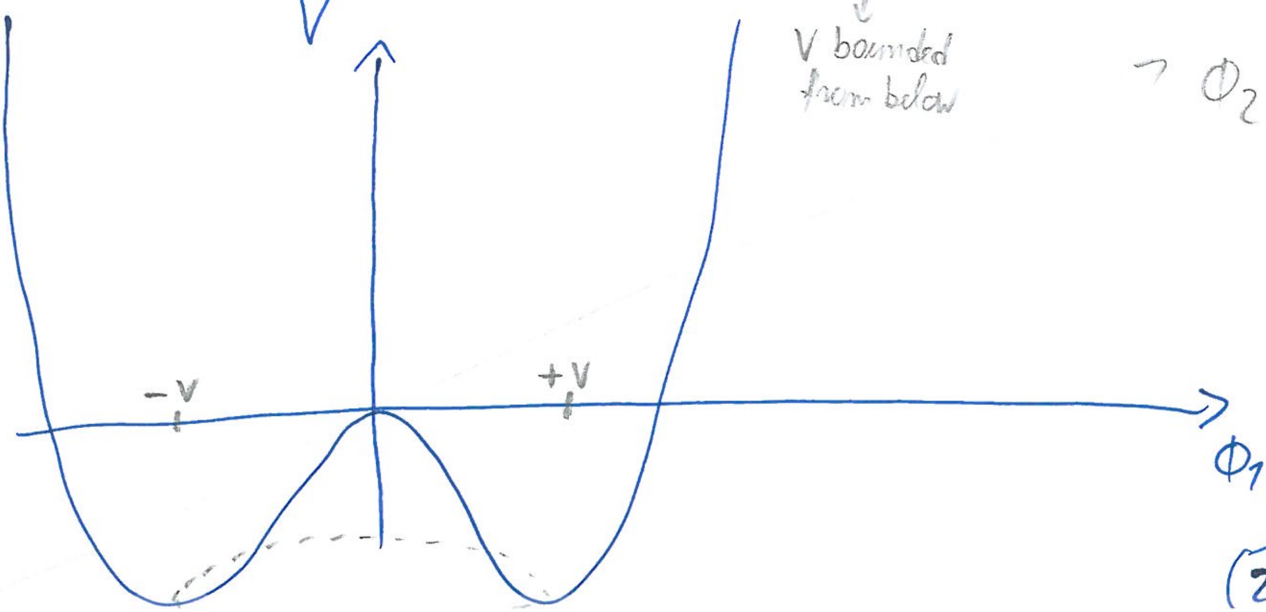
a) Global Case

$$(*) \quad \mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi)^* - \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2$$

with $\phi = \sqrt{\frac{1}{2}} (\phi_1 + i \phi_2)$ complex scalar field

Symmetry: $\phi \to e^{i\alpha} \phi$ global $[U(1)]$ on: $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \downarrow \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ $O(2)$

In potential: $V = \mu^2 \phi^* \phi + \lambda (\phi^* \phi)^2$, choose interesting case $\underbrace{\lambda > 0}_{\substack{\downarrow \\ V \text{ bounded} \\ \text{from below}}}$, $\mu^2 < 0$



the minimum of V is circle of radius v, with

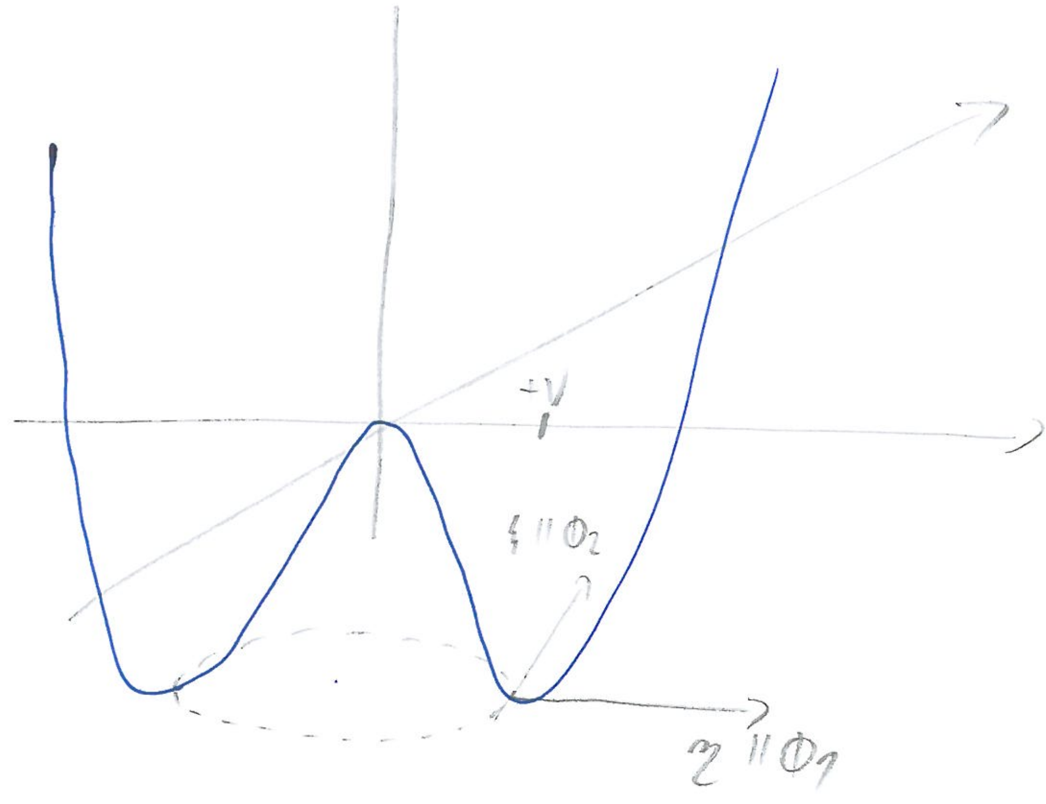
$$v^2 = \Phi_1^2 + \Phi_2^2 = -\mu^2/\lambda$$

we can only perform perturbation theory around minimum

w.l.o.g., choose $\Phi_1 = v \Rightarrow \Phi(x) = \sqrt{\tfrac{1}{2}}\left(v + \eta(x) + i\,\xi(x)\right)$

$\Rightarrow$ we call the fluctuations around the Vacum expectation value $\langle \Phi \rangle = v$ our physical particles $\Rightarrow$ insert in $\mathcal{L}$

$\Leftrightarrow$ $U(1)$ Symmetry is gone!



$$\mathcal{L}' = \tfrac{1}{2}(\partial_\mu \xi)^2 + \tfrac{1}{2}(\partial_\mu \eta)^2 + \underbrace{\mu^2 \eta^2}_{-\frac{1}{2} m_\eta^2 \eta^2} + \text{cubic, quartic, const}$$

correct mass term!

no mass term for ξ (flat direction)

$\rightarrow$ Goldstone Boson $\leftrightarrow$ spontaneously broken global symmetry

$\rightarrow$ $\boxed{\text{Goldstone Theorem}}$ If there is a global symmetry generated by N generators, and the vacuum state is invariant under $M < N$ generators, then there are $N-M$ massless Goldstone bosons.

Example: $O(3)$ scalar theory: $\phi = \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \end{pmatrix}$ real scalars

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)(\partial^\mu \phi^i) - \mu^2 \phi^i \phi^i - \lambda (\phi^i \phi^i)^2$$

vacuum: $\phi^i \phi^i = v^2 = -\mu^2/2\lambda$

w.l.o.g. choose $\langle \phi \rangle = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$

this vacuum is invariant under the group that leaves the last entry invariant : $O(2)$

$O(3)$ has 3 generators, $O(2)$ has 1

$\Rightarrow$ 2 massless GBs

[easily checked by inserting $\begin{pmatrix} \xi_1 \\ \xi_2 \\ v+\xi_3 \end{pmatrix}$]

a) Pseudo-Goldstone Bosons

in first global example, add $-\varepsilon \Phi_1^2/2$ term to $\mathcal{L}$ (from page 20)

"explicit breaking"

$$V = \mu^2/2 \, (\Phi_1^2 + \Phi_2^2) + \frac{\lambda}{4} (\Phi_1^2 + \Phi_2^2)^2 + \varepsilon/2 \, \Phi_1^2$$

$$\frac{\partial V}{\partial \Phi_1} = \varepsilon \Phi_1 + \mu^2 \Phi_1 + \lambda \Phi_1 (\Phi_1^2 + \Phi_2^2)$$

Note: $\mu^2 = -\lambda(\Phi_1^2 + \Phi_2^2)$ for $\varepsilon = 0$

$$\frac{\partial V}{\partial \Phi_2} = \mu^2 \Phi_2 + \lambda \Phi_2 (\Phi_1^2 + \Phi_2^2)$$

solved: $\Phi_2 = 0$ and $\Phi_1^2 = -(\mu^2 + \varepsilon^2)/\lambda = v^2$

$\Rightarrow$ insert $\begin{cases} \Phi_1 = v + \eta \\ \Phi_2 = \xi \end{cases}$ in V; equivalent to $-\mathcal{L}$

$$\Rightarrow \cancel{m_\eta^2} = \mathcal{L} \, c + (\mu^2 + \varepsilon^2)\eta^2 + \underbrace{\frac{\varepsilon}{2} \xi^2}_{\text{mass (small) of Pseudo-Goldstone-Boson}} + \dots$$

$\Leftrightarrow$ pion in "chiral symmetry" $\leftrightarrow$ small mass of u, d