I 3) Non-Abelian Gauge Theories

$SU(N)$: group of unitary $N \times N$ matrices
with det $= +1$

$$\boxed{\mathcal{U} = e^{i\vec{\alpha}\vec{t}} \simeq 1 + i\vec{\alpha}\vec{t} = 1 + i\alpha^a t^a}$$

replaces now $e^{i\alpha}$

moreover: instead of $e^{i\alpha}\psi$ we have $\mathcal{U}\psi \Rightarrow \psi$ is now a N-component object!

α_a : infinitesimal parameters

t_a : "generators" of $SU(N)$

from $\mathcal{U}\mathcal{U}^\dagger = \mathbb{1}$ and $\det \mathcal{U} = +1$: $\boxed{\begin{array}{l} t_a^\dagger = t_a \\ \mathrm{Tr}\{t_a\} = 0 \end{array}}$

$SU(N)$ has $N^2 - 1$ generators, forming a Lie-Algebra

$$[t^a, t^b] = i f^{abc} t^c$$

; antisymmetric structure constants

($U(1)$ has the generator 1)

Examples: •) $SU(2)$:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli matrices

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2} \right] = i \, \varepsilon_{ijk} \frac{\sigma_k}{2}$$

•) $SU(3)$:

$$\left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2} \right] = i f_{ijk} \frac{\lambda_k}{2}$$ 8 Gell-Mann matrices

<u>Comments</u>: •) structure ~~functions~~ constants can be trougt of as representation of algebra ; here generators are represented by $(N^2-1) \times (N^2-1)$ matrices:

$$(t^a)_{jk} = -i f_{ajk}$$ „<u>adjoint representation</u>"

N^2-1 dim. represent.

•) there are ~~[illegible]~~ higher-dimensional representations of $SU(N)$; the N-dimensional is called <u>fundamental representation</u> (all higher repr. formed by it)

e.g. $2t_1^{(3)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} ; 2t_2^{(3)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$

$$2t_3^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

→ apply $SU(N)$ trafo to Dirac-equation

↪ need N component object

~~[illegible]~~

e.g. $L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}$, $Q = \begin{pmatrix} u \\ d \end{pmatrix}$ $\quad SU(2)_L$

$q = \begin{pmatrix} u_r \\ u_g \\ u_b \end{pmatrix}$ $\quad SU(3)_c$

trafo: $\boxed{\psi_i \to \psi_i' = U_{ij}\,\psi_j}$ with $U_{ij} \simeq (1 - i\theta^a t^a)_{ij}$

$i = 1, \dots, N$

or: $\psi \to \psi' = U\psi\ ;\ \bar\psi \to \bar\psi' = \bar\psi U^\dagger = \bar\psi U^{-1}$

gauge trafo with covariant derivative:

$\mathcal{L}' = \bar\psi U^\dagger (\not{D}' - m) U\psi \implies U^\dagger D_\mu' U \overset{!}{=} D_\mu$

or $D_\mu' U\psi = U D_\mu \psi$

~~proof: D_μ ...~~

$$\implies \boxed{\begin{aligned} D_\mu &= \partial_\mu - ig A_\mu^a t^a = \partial_\mu - ig \vec{A}_\mu \vec{t} \equiv \partial_\mu - ig \tilde{A}_\mu \\ &\text{with } \tilde{A}_\mu \to \tilde{A}_\mu' = \frac{-i}{g} (\partial_\mu U) U^{-1} + U \tilde{A}_\mu U^{-1} \end{aligned}}$$

[cf. $A_\mu \to A_\mu' = A_\mu + \frac{1}{e}\partial_\mu \alpha$ for $U(1)$]

<u>proof:</u> $D'_\mu \psi' = \left[\partial_\mu - ig\left(\frac{-i}{g}(\partial_\mu \mathcal{U})\mathcal{U}^{-1} + \mathcal{U}\tilde{A}_\mu \mathcal{U}^{-1}\right)\right]\mathcal{U}\psi$

$$= \underline{(\partial_\mu \mathcal{U})\psi} + \mathcal{U}(\partial_\mu \psi) - \underline{(\partial_\mu \mathcal{U})\psi} - ig\,\mathcal{U}\tilde{A}_\mu \psi$$

$$= \mathcal{U}(\partial_\mu - ig\tilde{A}_\mu)\psi = \mathcal{U} D_\mu \psi \quad ☺$$

$\Rightarrow$ $N^2 - 1$ gauge fields, one for each generator

•) <u>infinitesimal trafo:</u>

$$A_\mu^a t^a \to (1 - i\theta^b t^b) A_\mu^a t^a (1 + \theta^b t^b)$$

$$- \frac{i}{g}\left[\partial_\mu (1 - i\theta^a t^a)\right](1 + i\theta^b t^b)$$

$$= A_\mu^a t^a + i\theta^b A_\mu^a [t^a, t^b] - \frac{1}{g}(\partial_\mu \theta^a) t^a$$

$$= A_\mu^a t^a - \frac{1}{g}(\partial_\mu \theta^a) t^a + f^{abc}\theta^b A_\mu^c t^a$$

$\left[\text{cf. } A_\mu' = A_\mu + \frac{1}{e}(\partial_\mu \alpha)\right]$

$\delta\tilde{A} \sim f_{abc} A$
adjoint rep.

•) kinetic terms

$$F_{\mu\nu} = \frac{i}{g}\left[D_\mu, D_\nu\right] = F^a_{\mu\nu}\, t^a$$

$$= \dots = \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \underbrace{g f_{abc} A_\mu^b A_\nu^c}_{\text{new term} \leftrightarrow \text{non-Abelian}}\right) t^a$$

is it gauge invariant?

$$\left[D'_\mu, D'_\nu\right] = \left[\mathcal{U} D_\mu \mathcal{U}^{-1}, \mathcal{U} D_\nu \mathcal{U}^{-1}\right] = \mathcal{U}\left[D_\mu, D_\nu\right]\mathcal{U}^{-1}$$

$\Rightarrow$ $F_{\mu\nu}$ alone <u>not</u> invariant ($\leftrightarrow$ only for Abelian)

$\Rightarrow$ use $\mathrm{Tr}\left\{F_{\mu\nu} F^{\mu\nu}\right\}$ as kinetic term

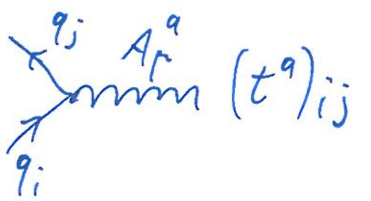
$(\delta F^a_{\mu\nu} = -f_{abc} F^b_{\mu\nu} t^c)$

<u>Note:</u> •) $m_W^2 A_\mu^a A^{\mu a}$ still forbidden

•) A_μ^a : $N^2 - 1$ gauge bosons

•) $F_{\mu\nu} F^{\mu\nu} \sim (\partial_\mu A + g A^2)^2 \sim g A^3 + g^2 A^4$

self interactions!

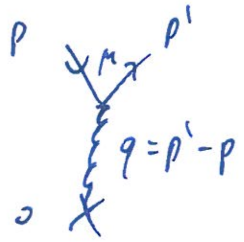
•) 

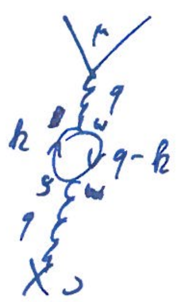
$\Rightarrow$ Feynman rules include the generators

$\mathcal{L} \ni ig\, \bar{\psi} \gamma^\mu t^a \psi A_\mu^a$

Gauge boson self-interactions and running couplings

simplest example QED





„loop correction" to tree level scattering of electrons off a potential A_μ

$$\mathcal{M}_{tree} \propto e\, \bar{u}(p')\gamma^\mu u(p) \frac{g_{\mu\nu}}{q^2} j^\nu(q)$$

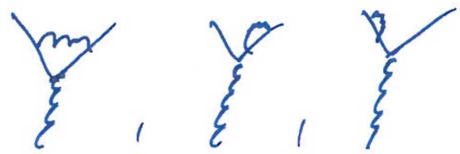
$$\mathcal{M}_{loop} \propto (-1)\, e\, \bar{u}(p')\gamma^\mu u(p) \frac{-i g_{\mu\omega}}{q^2} \int \frac{d^4k}{(2\pi)^4} \times$$

$$e\,(\gamma^\omega)_{\alpha\beta} \frac{(\not{k}+m)_{\beta\delta}}{k^2-m^2}\, e\,(\gamma^\rho)_{\delta\varepsilon} \frac{(\not{k}-\not{q}+m)_{\varepsilon\alpha}}{(k-q)^2-m^2} \frac{g_{\rho\nu}}{q^2} j^\nu(q)$$

$\Rightarrow \mathrm{Tr}\{\ldots\}$

diagram diverges: $\int \frac{d^4k}{k^2}$ turns out: only logarithmically (+ gauge invariance)

„Ward identities"

+  divergences cancel

$\Rightarrow$ same correction for e, μ, τ

effect of „vacuum polarization"

$$\frac{-i g_{\mu\nu}}{q^2} + \left(\frac{-i g_{\mu\omega}}{q^2}\right) I^{\omega\rho} \left(-i \frac{g_{\rho\nu}}{q^2}\right)$$

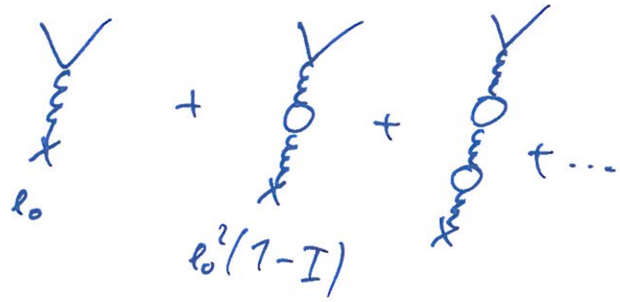
with $I^{\omega\rho} = -i g^{\omega\rho} q^2 I(q^2)$

where $I(q^2) = \frac{\alpha}{3\pi} \log M^2/_{-q^2} = \frac{\alpha}{3\pi} \log \frac{M^2}{Q^2}$

here limit is $-q^2 \gg m^2$; <u>cutoff</u> M^2 has been introduced

effectively, charge is modified: $e^2 \to e^2(1-I)$

<u>but</u>: experiment measures



e_0: term in $\mathcal{L}$ „bare charge"

$\Rightarrow$ define $\boxed{\alpha = \alpha_0(1-I)}$ „renormalized charge"

α: finite (+small)
α_0: „infinite"

$I = I(Q^2) \Rightarrow$ Effect: $\alpha = \alpha(Q^2)$ running coupling!

runs logarithmically $\Leftrightarrow$ compare at different scales use $\alpha(Q^2) = \alpha_0 (1 - I(Q^2))$; $I = \frac{\alpha_0}{3\pi} \log M^2/Q^2$

measure at reference scale μ^2

$$\alpha_0 = \frac{\alpha(\mu^2)}{1 - \frac{\alpha_0}{3\pi}\log M^2/\mu^2} \simeq \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi}\log M^2/\mu^2}$$

(correct to order α_0^2)

$$\Rightarrow \alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi}\log M^2/\mu^2}\left(1 - \frac{\alpha_0}{3\pi}\log M^2/Q^2\right)$$

$$\simeq \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi}\log M^2/\mu^2}\left(1 - \frac{\alpha(\mu^2)}{3\pi}\log M^2/Q^2\right)$$

$$\simeq \alpha(\mu^2)\left[1 + \frac{\alpha(\mu^2)}{3\pi}\underbrace{\left(\log M^2/\mu^2 - \log M^2/Q^2\right)}_{\log Q^2/\mu^2}\right]$$

$$\Rightarrow \boxed{\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi}\log Q^2/\mu^2}}$$

[replace $\frac{\alpha}{3\pi}$ with $\frac{\alpha N_f}{3\pi}$; number of generations (3)]

$\alpha \nearrow$ for $Q^2 \nearrow$

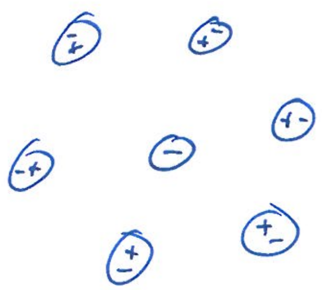
generic for any $U(1)$

"Landau pole"

$$\alpha(Q^2 \simeq 0) \simeq \frac{1}{137}$$

$$\downarrow$$

$$\alpha(Q^2 = M_Z^2) \simeq \frac{1}{128}$$



$Q^2 \nearrow$ one sees more from charge

$Q^2 \searrow$ sees less "screening"