Exercises to "Standard Model of Particle Physics"

Summer 2013

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Exercise 15: Goldstone theorem [**20 Points**]

The Goldstone–Theorem states: every generator of a continuous global symmetry, which does not annihilate the vacuum, corresponds to a massless scalar particle.
As "proof", consider the Lagrangian $\mathcal{L}(\Phi_i)$ with real scalar fields Φ_i, invariant under the global transformation $\vec{\Phi} \to \exp(i\theta^a T^a)\,\vec{\Phi}$.

1. Which properties have the iT^a, if the Φ_i are real and if $\Phi^T\Phi$ is invariant under the transformation?

2. Show that from the conservation of the Noether current it follows that:

$$\frac{\partial \mathcal{L}}{\partial \Phi_i}\,(iT^a)_{ij}\,\Phi_j + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \Phi_i)}\,(iT^a)_{ij}\,\partial^\mu \Phi_j = 0.$$

3. Let $\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\Phi})^T\,(\partial^\mu \vec{\Phi}) - V(\vec{\Phi})$. Show with the help of the first two results that

$$\frac{\partial V}{\partial \Phi_i}\,(iT^a)_{ij}\,\Phi_j = 0.$$

4. Let $\vec{v}$ be the minimum of $V(\vec{\Phi})$. Show with the results from above that $M^2\,(iT^a)\,\vec{v} = 0$, where M^2_{ij} is given by

$$(M^2)_{ij} := \left.\frac{\partial^2 V}{\partial \Phi_i\,\partial \Phi_j}\right|_{\vec{\Phi}=\vec{v}}$$

5. The matrix M^2 is interpreted as a mass matrix. The broken vacuum $\vec{v}$ is in general not invariant under the transformations $\exp(i\theta^a T^a)$. If $T^a\vec{v} = 0$, one has a *Wigner–Weyl realization* of the symmetry, if $T^a\vec{v} \neq 0$ one has a realization à la *Nambu–Goldstone*. With the result from the last point it follows that for all a with $T^a\vec{v} \neq 0$ the mass matrix has an eigenvector with eigenvalue 0, i.e. the scalar field $\phi_i(iT^a)_{ij}v_j$ (with $\phi_i = \Phi_i - v_i$) is massless.
Consider e.g. the Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\Phi})^T(\partial^\mu \vec{\Phi}) - V(\vec{\Phi}^T\vec{\Phi})$ with the potential $V(\vec{\Phi}^T\vec{\Phi}) = \frac{1}{2}\mu^2\vec{\Phi}^T\vec{\Phi} + \frac{1}{4}\lambda(\vec{\Phi}^T\vec{\Phi})^2$, where $\vec{\Phi}^T = (\phi_1, \phi_2, \phi_3)$ is a triplet of real and scalar particles. The coupling obeys $\lambda > 0$.

(i) Show that $\mathcal{L}$ is invariant under a $SU(2)$ transformation $\vec{\Phi} \to \exp(i\theta^a T^a)\,\vec{\Phi}$. The 3 generators T^j of $SU(2)$ are written here as $(T^j)_{kl} = -i\epsilon_{jkl}$ (this is called the adjoint representation of $SU(2)$).

(ii) For $\mu^2 < 0$ the potential has a minimum for $\vec{v}^T = (0, 0, v)$. How are v, μ^2 and λ connected? Show that the vacuum state $\vec{v}$ is invariant under the transformation with T^3, but not with T^1 and T^2. That is, the vacuums state does *not* possess the symmetry of the Lagrangian (spontaneous symmetry breaking). Why do we speak in this case of spontaneous symmetry breaking from $SU(2)$ to $U(1)$?

(iii) The scalar fields will be expanded around the minimum: $\vec{\Phi}^T \equiv (\phi_1, \phi_2, v + \phi_3)$. Show by inserting into the above Lagrangian that there is one massive scalar fields and 2 massless Goldstone bosons. What is the mass of ϕ_3? Convince yourself that the matrix M^2 from part 4 is really the mass matrix.

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Tutorials homepage: `http://www.mpi-hd.mpg.de/manitop/StandardModel/exercise.html`

Hand-in of sheet:
during lecture on 17.6.

Discussion of sheet:
Thursday, 20.6. 2.15 pm, INF 227 SR 2.402
Friday, 21.6. 2.15 pm, INF 227 SR 1.403