Additional Material and Theoretical Details to „QED and basic processes"

1) Processes with external photons

Recall: $\psi(x) = -e \int d^4x' \, K(x-x') A(x') \psi(x')$

can be solved iteratively:

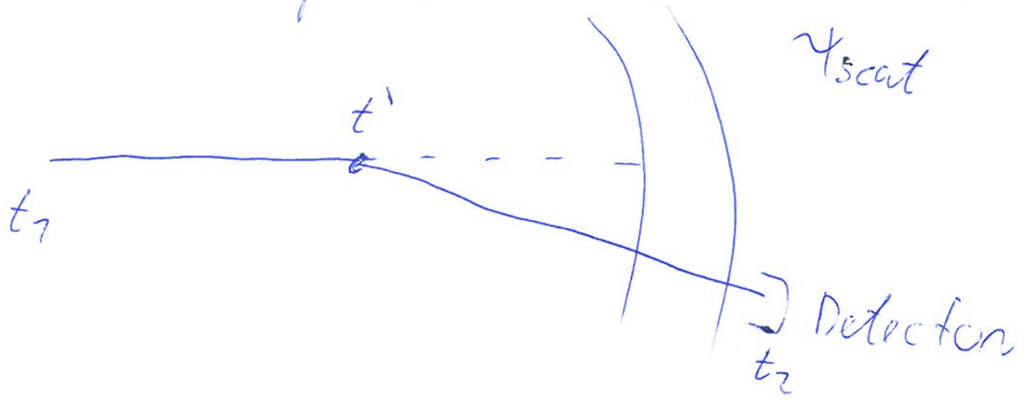
$$\psi^{(0)} = \Phi(x) \quad [\text{free solution: } \Phi]$$

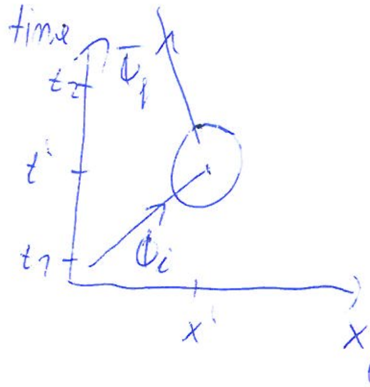
$$\psi^{(1)} = \Phi - e \int d^4x' \, K(x-x') A(x') \Phi(x')$$

$$\psi^{(2)} = \Phi(x) - e \int d^4x' \, K(x-x') A(x') \Phi(x')$$

$$+ e^2 \int d^4x' d^4x'' \, K(x-x'') A(x'') K(x''-x') A(x') \Phi(x')$$

let us consider the third term of $\psi^{(2)}(x)$ for our example from April 30, page 36:





$$S^{(2)}_{fi} = \int d^3x_2\ \Phi_f^\dagger(x_2)\Psi_{scat}(x_2)$$ ($\leftrightarrow e^-$ interacts twice with potential)

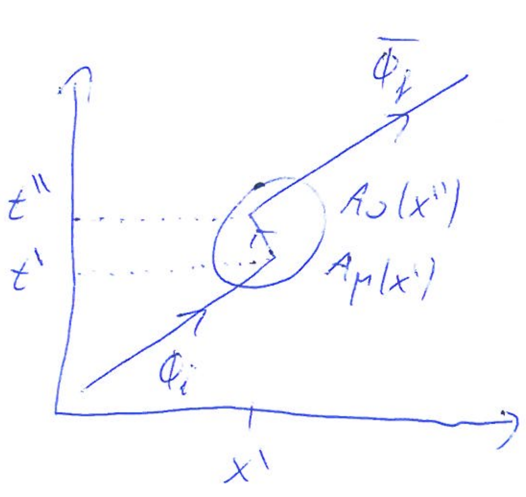
$$= e^2 \int d^4x'\, d^4x''\, \underbrace{d^3x_2\ \Phi_f^\dagger(x_2)\, K(x_2 - x'')}_{-i\,\overline{\Phi}_f(x'')}\, \not{A}(x'')\, K(x''-x')\, \not{A}(x')\, \Phi_i(x')$$

$$= -ie^2 \int d^4x'\, d^4x''\ \overline{\Phi}_f(x'')\, \underset{}{\overset{A_\nu\gamma^\nu}{\not{A}}}(x'')\, K(x''-x')\, \overset{A_\mu\gamma^\mu}{\not{A}}(x')\, \Phi_i(x')$$

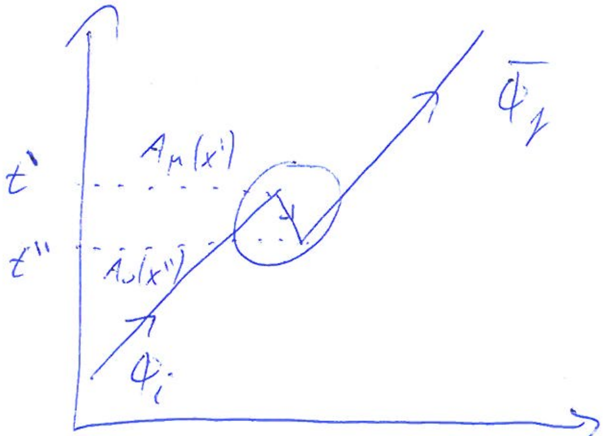
we measure initial and final state, but we do not know whether $t'' > t'$ or $t'' < t'$

$t'' > t'$ $\downarrow$ e^- with $p^0 > 0$ goes into future

$t'' < t'$ $\downarrow$ e^- with $p^0 < 0$ goes into past



e^- scatters twice



at x'': e^+e^- pair created; e^+ annihilates with e^- at x'

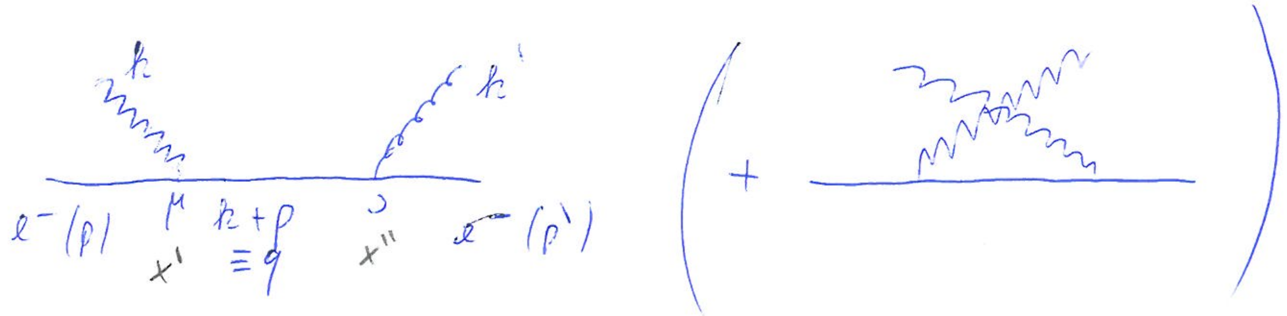
x' Integration: $(2\pi)^4\,\delta(q-(k+p))$

x'' " $:(2\pi)^4\,\delta(q-(k'+p'))$

$\hookrightarrow$ q - Integration: $\delta(k+p-k'-p')$

$\Rightarrow$ if in a diagram in an internal line the momentum is <u>not</u> fixed by Energy - conservation: $\int \frac{d^4q}{(2\pi)^4}$

On Compton scattering:



$$S_{fi} = -ie^2 \int d^4x' d^4x'' \, \bar{\Phi}_f(x'') \Big[\not{A}'^*(x'') \, K(x''-x') \, \not{A}(x')$$

$$+ \not{A}(x'') \, K(x''-x') \, \not{A}'^*(x') \Big] \Phi_i(x')$$

$$A_\mu(x) = \varepsilon_\mu e^{-ikx} \quad ; \quad A'_\mu(x) = \varepsilon'_\mu e^{-ik'x}$$

and A_μ^* for outgoing photon

[recall $\sqrt{\frac{1}{2E_k}}$ factor from normalization in A, Φ]

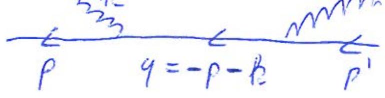
again: $\Phi_i(x') = u(p) e^{-ipx'}$ $\qquad \bar{\Phi}_f(x'') = \bar{u}(p') e^{ip'x''}$

$\Rightarrow$ 1st term in S_{fi}:

$$-ie^2 \int d^4x' d^4x'' \, \bar{u}(p') \, \varepsilon'^*_\nu \gamma^\nu \frac{d^4q}{(2\pi)^4} \frac{\not{q}+m}{q^2-m^2} e^{-iq(x''-x')} \times$$

$$\varepsilon_\mu \gamma^\mu u(p) \, e^{+ik'x''} e^{-ikx'} e^{ip'x''} e^{-ipx'}$$

Remark: positrons:

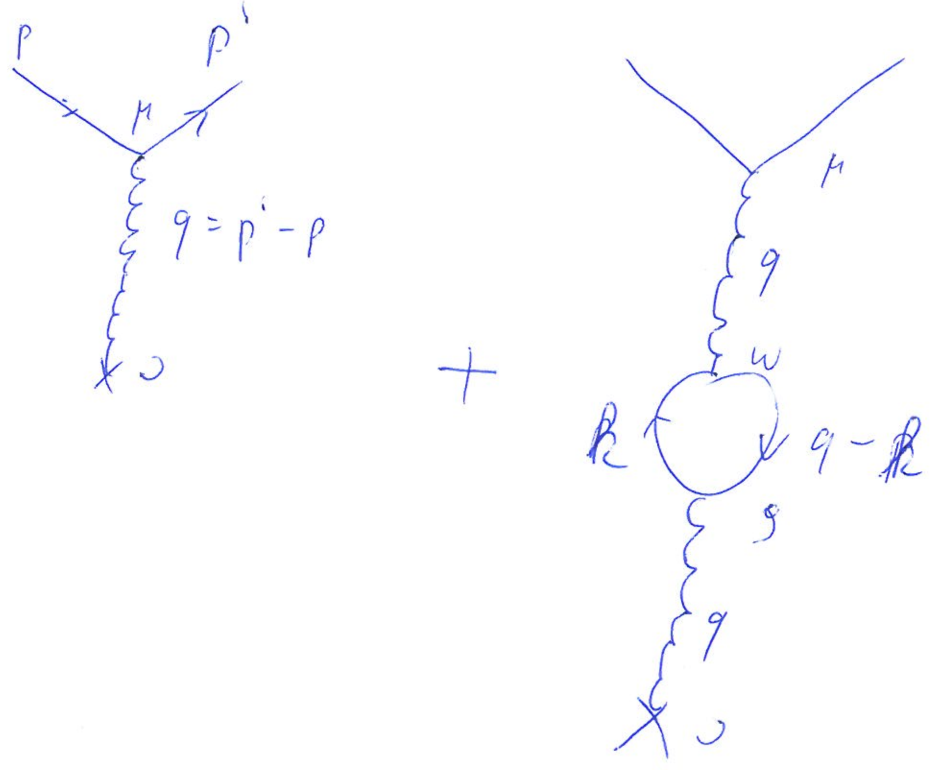


$S \propto \bar{v}(p) \ldots v(p')$

momentum flow opposite to arrows on external, not internal lines

2) Running coupling in QED

Correction to scattering off a potential



loop – correction
vacuum polarization

without the loop: $S_{fi} = -ie \int d^4x\, \bar{u}(p')\, \not{A}(x)\, u(p)\, e^{-iqx}$

$$= -ie\, \bar{u}(p')\, \not{A}(q)\, u(p)$$

$$\equiv -ie\, \bar{u}(p')\, \gamma^\mu u(p)\, \frac{-ig_{\mu\nu}}{q^2} \left(-i j^\nu(q)\right)$$

(to make it look familiar...)

static source:
$j^0(\vec{x}) = Ze\,\delta(\vec{x})$
$\vec{j} = 0$
or $A^\mu = \left(\frac{Ze}{4\pi|\vec{x}|}, \vec{0}\right)$
not important here...

leads to $\left(\frac{d\sigma}{d\Omega}\right)_0$, not important here now the loop diagram:

Fermion loop: never mind...

$$-i\Omega = (-1)\; ie\, \bar{u}_f \gamma^\mu u_i \; \frac{-i g_{\mu\omega}}{q^2} \int \frac{d^4 k}{(2\pi)^4}$$

$$ie(\gamma^\omega)_{\alpha\beta} \frac{i(\not{k}+m)_{\beta\gamma}}{k^2-m^2}\; ie(\gamma^\rho)_{\gamma\delta} \frac{i(\not{k}-\not{q}+m)_{\delta\alpha}}{(k-q)^2-m^2}$$

$$\times \frac{-i g_{\rho\nu}}{q^2} \left(-i j^\nu(q)\right)$$

- Note the trace $\mathrm{Tr}\left\{ \gamma^\omega (\not{p}+m)\gamma^\rho (\not{p}-\not{q}+m)\right\}$

- effect of loop: modification of propagator of order α

$$-i\frac{g_{\mu\nu}}{q^2} + \left(-i\frac{g_{\mu\omega}}{q^2}\right) I^{\omega\rho} \left(-i\frac{g_{\rho\nu}}{q^2}\right)$$

$I^{\omega\rho}$ diverges logarithmically...

$\Rightarrow$ introduce cutoff M^2

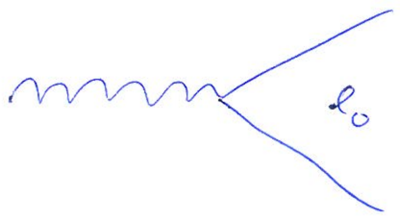
$$\Rightarrow I^{\omega\rho} = -i g^{\omega\rho} q^2 I(q^2)$$

with $I(q^2) = \begin{cases} \frac{\alpha}{3\pi}\log\frac{M^2}{m^2} + \frac{\alpha}{15\pi}\frac{q^2}{m^2} & (-q^2 \ll m^2) \\ \frac{\alpha}{3\pi}\log\frac{M^2}{-q^2} & (-q^2 \gg m^2) \end{cases}$

$$\Rightarrow -i\sqrt{2}\,\alpha\, e\left(1-\underbrace{\frac{\alpha}{3\pi}\log\frac{M^2}{-q^2}}_{I}\right)\frac{1}{q^2}\,Ze \qquad \to \infty\ !$$

Is this a problem? No!

Experimentalist does not measure the <u>bare charge e_0</u>



he measures:

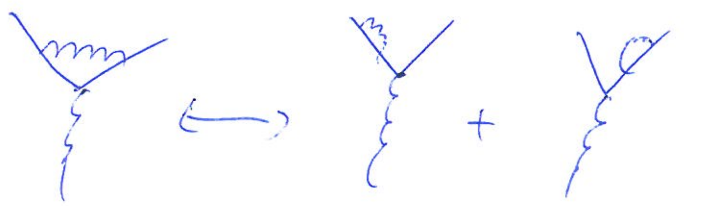
$+ \ldots$

$\Rightarrow$ call $e = e_0\,(1-I)^{1/2}$ with finite e;

relation between e and e_0 depends on energy:

$$\Rightarrow \quad \boxed{e = e_0\left(1-I/2\right)\Big|_{Q^2=\mu^2}} \qquad (\text{with } Q^2=-q^2>0)$$

Remark: vertex corrections cancel each other:



(Ward identity)

$\to$ gauge invariance

$\Rightarrow I$ independent of m $\Rightarrow$ charge redefinition independent!

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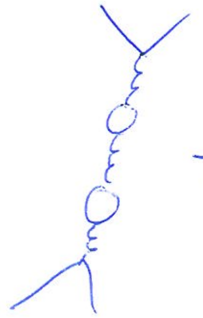
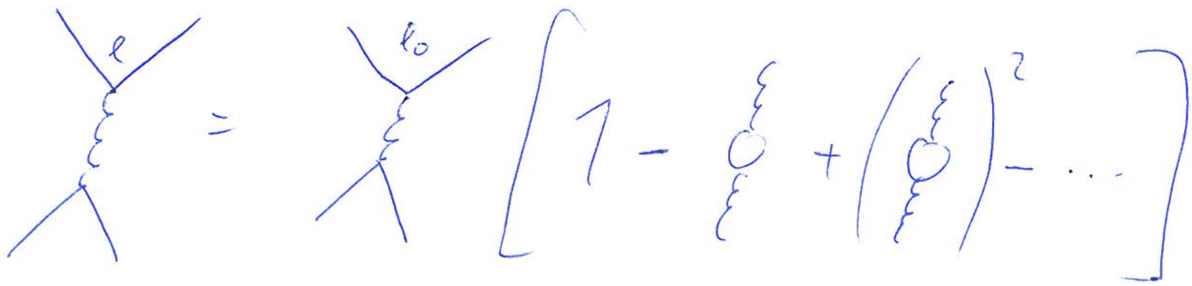
$\Rightarrow$ higher order terms from  $+ \dots$

or



$$\Rightarrow e^2 = e_0^2\,(1 - I + I^2 - I^3 + \dots)\Big|_{Q^2=\mu^2}$$

$$\boxed{\alpha(Q^2) = \frac{\alpha_0}{1+I}}$$ (geometric series)

$$\boxed{= \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi}\log Q^2/\mu^2}}$$

now we use that from $\alpha = \alpha_0(1 - I(\mu^2))$ follows:

$$\alpha_0 = \alpha(\mu^2)\,(1 + I(\mu^2))$$

$$= \alpha(\mu^2)\left(1 + \frac{\alpha(\mu^2)}{3\pi}\log \mu^2/\mu^2\right)$$

(perturbative, correct to the order we are working with)

$$\Rightarrow \alpha(Q^2) = \frac{\alpha(\mu^2)\left(1 + \frac{\alpha(\mu^2)}{3\pi}\log M^2/\mu^2\right)}{1 - \frac{\alpha(\mu^2)}{3\pi}\log Q^2/\mu^2}$$

$$\simeq \alpha(\mu^2)\left(1 + \frac{\alpha(\mu^2)}{3\pi}\log M^2/\mu^2\right)\left(1 + \frac{\alpha(\mu^2)}{3\pi}\log Q^2/\mu^2\right)$$

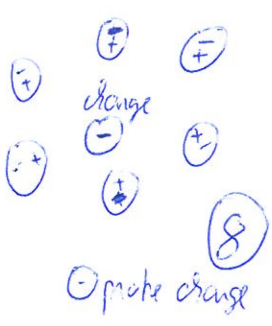
$$\simeq \alpha(\mu^2)\left[1 + \frac{\alpha(\mu^2)}{3\pi}\underbrace{\left(\log M^2/\mu^2 + \log Q^2/\mu^2\right)}_{= \log Q^2/\mu^2}\right]$$

$$\Rightarrow \boxed{\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi}\log Q^2/\mu^2}}$$

! RUNNING COUPLING !

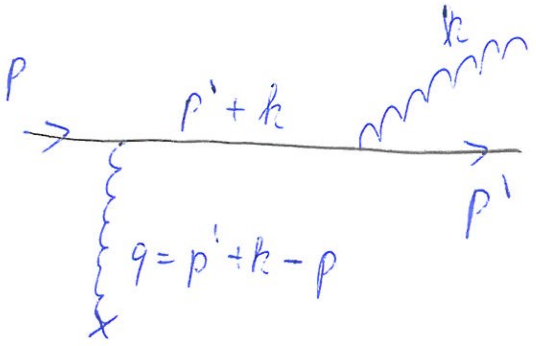
the higher Q^2, the more charge the photon sees;
„screening" ; $\alpha \nearrow \infty$ for $Q^2 \nearrow \infty$

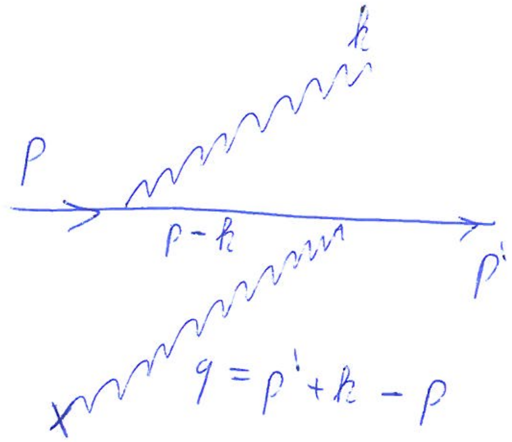
$\alpha(Q^2=0) \simeq \frac{1}{137}$; $\alpha(Q^2 = M_Z^2) \simeq \frac{1}{128}$



3) Bremsstrahlung

(braking radiation)





radiation emitted by an electron as it moves past a nucleus (or any charge)

$$-i\mathcal{M} = -e^2\, \bar{u}(p')\left[\not{\varepsilon}^* \frac{\not{p}' + \not{k} + m}{(p'+k)^2 - m^2} \not{A}(q) + \right.$$

$$\left. \not{A}(q) \frac{\not{p} - \not{k} + m}{(p-k)^2 - m^2} \not{\varepsilon}^* \right] u(p)$$

$$\boxed{\frac{d\sigma}{d\Omega'} = \frac{1}{(2\pi)^5\, 2\omega} \frac{|\vec{p}\,'|}{|\vec{p}|} \overline{|\mathcal{M}|^2}\, d^3k}$$ with $k^\mu = (\omega, \vec{k})$

of interest here: „soft photon" $\omega \simeq 0$

$\Rightarrow$ neglect k in numerator; $q \simeq p' - p$; $|\vec{p}\,'| \simeq |\vec{p}|$

use that $\bar{u}(p')\,\not\varepsilon^*(\not p'+m)\not A(q)\,u(p)$

$$= \bar{u}(p')\,\gamma_\mu(\gamma_\rho p'^\rho + m)\,\gamma_\nu\, u(p) \quad \varepsilon^{*\mu} A^\nu$$

$$= \bar{u}(p')\left[(-\gamma_\rho\gamma_\mu + 2g_{\mu\rho})\,p'^\rho + m\gamma_\mu\right]\gamma_\nu\, u(p) \quad \varepsilon^{*\mu} A^\nu$$

$$= \bar{u}(p')\left[-m\gamma_\mu + 2p'_\mu + m\gamma_\mu\right]\gamma_\nu\, u(p) \quad \varepsilon^{*\mu} A^\nu$$

$$= 2\bar{u}(p')\not A\, u(p)\,(p'\cdot\varepsilon^*)$$

$$\Rightarrow \mathcal{M} = -e\,\mathcal{M}_0\left[\frac{p'\cdot\varepsilon^*}{p'\cdot k} - \frac{p\cdot\varepsilon^*}{p\cdot k}\right]$$

with $\mathcal{M}_0 = ie\,\bar{u}(p')\not A(q)\,u(p)$ the amplitude for scattering without photon emission (page 4)

Problem: Integration over k creates divergence!

<u>"infrared divergence"</u> ($\rightarrow$ radiating off a massless particle)

$$\frac{1}{(p-k)^2-m^2} \sim \frac{1}{-2pk} \sim \infty \text{ for } k\to 0$$

OR: $d\sigma \sim \frac{1}{k}\,d^3k\,\frac{1}{k^2} \sim \frac{dk}{k}$ $\rightarrow$ Prob. to emit zero energy γ: ∞

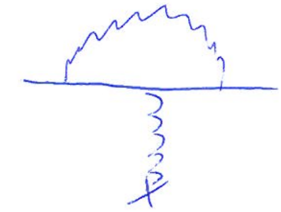
finite result possible when giving photon a small mass μ: (e.g. Kaku: Quantum Field Theory)

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_B = \left(\frac{d\sigma}{d\Omega}\right)_0 \frac{\alpha}{\pi} \ln\frac{E^2}{\mu^2} \log\frac{-q^2}{m^2}}$$

nicely factorizes into $\left(\frac{d\sigma}{d\Omega}\right)_0$

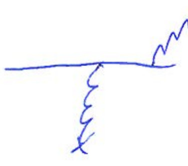
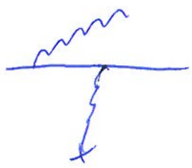
times $\underline{\mathcal{O}(\alpha) \text{ correction}}$.

For consistency: should consider



i.e. $\mathcal{O}(\alpha)$ correction to $\left(\frac{d\sigma}{d\Omega}\right)_0$

Reason: there is always energy resolution in experiment, and some soft photons of

 + 

will escape undetected

$\leftrightarrow$ measurement process in QM

One can show that:

$$\left(\frac{d\sigma}{d\Omega}\right)_V = \left(\frac{d\sigma}{d\Omega}\right)_0 \left(1 - \frac{\alpha}{\pi} \log \frac{-q^2}{m^2} \log \frac{-q^2}{\mu^2}\right)$$

(i.e. also diverges unless photon mass is introduced)

add this to $\left(\frac{d\sigma}{d\Omega}\right)_B$: μ^2 drops out ☺

"Bloch-Nordsieck-Theorem"

Infrared divergences from phase space integrals cancel divergences of loop corrections

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$\leftrightarrow$ Renormalization, beyond the scope of this lecture
(ultraviolett divergences)

deeply connected to gauge theories