Recap.: $(i\not\partial - m)\psi(x) = -e \not A(x)\psi(x) \quad (*)$

How to describe and propagate fermions and photons

$$(i\not\partial - m)K(x,x') = \delta(x-x')$$

Green's function ($\leftarrow$ inh. diff. eq. with boundary conditions)

if K known: $\psi(x) = \underbrace{\phi(x)}_{\text{free solution}} - e\int d^4x'\, K(x-x')\not A(x')\psi(x')$

solves $(*)$!

approx. approach ($\alpha \ll 1$): $\psi^{(1)}(x) = \phi(x) - e\int d^4x'\, K(x-x')\not A(x')\phi(x')$

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note: $\psi(x) = \sum_s \int \frac{d^3p}{[(2\pi)^3 2E]^{1/2}}\left[a_s(p)u_s(p)e^{-ipx} + b_s^*(p)v_s(p)e^{ipx}\right]$

such that in $j^\mu = \bar\psi\gamma^\mu\psi$ the zeroth component

$\int \psi^+\psi\, d^3x = 1$ ($\leftrightarrow$ probability density)

Also: actually $\int \frac{d^3p}{[(2\pi)^3 2EV]^{1/2}}$ with $V = 1$

$\Rightarrow [\psi] = E^{3/2} \leftrightarrow [\bar\psi(i\not\partial - m)\psi] = E^4$ ✓

Propagator

turns out to be useful to calculate FT of K

$$K(x-x') = \frac{1}{(2\pi)^4}\int d^4p\, \tilde{K}(p)\, e^{-ip(x-x')}$$

insert in definition of $K(x-x')$:

$$(i\not\partial - m)\, K(x-x') = \frac{1}{(2\pi)^4}\int d^4p\, (\not p - m)\, \tilde{K}(p)\, e^{-ip(x-x')}$$

$$\overset{!}{=} \delta(x-x')$$

$$\Rightarrow (\not p - m)\, \tilde{K}(p) = \mathbb{1}$$

$$\Rightarrow \boxed{\tilde{K}(p) = \frac{1}{\not p - m} = \frac{\not p + m}{p^2 - m^2}}$$

Elektron - propagator

(only defined for virtual electrons) $p^2 \neq m^2$

$$\Rightarrow K(x-x') = \frac{1}{(2\pi)^4}\int d^3p\, e^{-i\vec{p}(\vec{x}-\vec{x}')}\int_{-\infty}^{\infty} dp_0\, \frac{e^{-ip_0(t-t')}(\not p + m)}{(p_0 - E)(p_0 + E)}$$

with p_0 independent of $E \equiv \sqrt{\vec{p}^2 + m^2}$ ($\leftrightarrow$ virtual electrons)

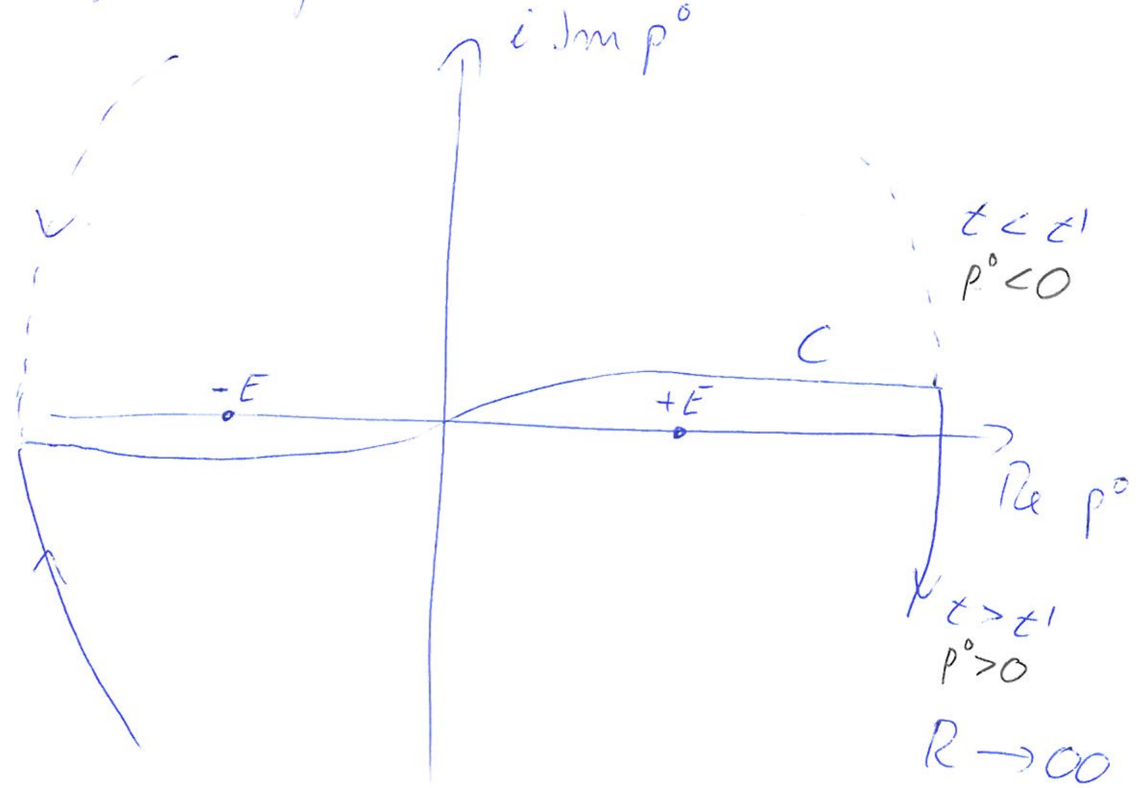
Integral does not seem to converge ...

Reason: boundary conditions not specified!

Demand that solutions with $t > t'$ have $p^0 > 0$

$t < t'$ $\quad p^0 < 0$

$\leftrightarrow$ change integration contour to C



use residue theorem: $\oint_C f(z)\,dz = 2\pi i \lim_{a \to a_k} (a - a_k) f(a_k)$

closed path

a_k: first order poles

for $t > t'$: $\int dp^0 \frac{1}{p^0 - E} \underbrace{\frac{(\not p + m)\, e^{-ip^0(t-t')}}{p^0 + E}}_{\tilde f(p)}$

$= -2\pi i\, \tilde f(p^0 = E)$

$$\Rightarrow \boxed{K(x-x') = i(2\pi)^{-3}\int d^3p\, e^{-i[E(t-t')-\vec{p}(\vec{x}-\vec{x}')]}\, \frac{\gamma_0 E - \vec{\gamma}\vec{p} + m}{2E}}$$

for $t > t'$

in analogy:

$$\boxed{K(x-x') = -i(2\pi)^{-3}\int d^3p\, e^{i[E(t-t')+\vec{p}(\vec{x}-\vec{x}')]}\, \frac{-\gamma_0 E - \vec{\gamma}\vec{p} + m}{2E}}$$

[note $E > 0$; $p^0 = \pm E$]

for $t' > t$

properties (check at home!): for positive p^0: $(\leftrightarrow \Phi = u\, e^{-ipx})$

a) $$i\int d^3x'\, K(x-x')\,\gamma^0\,\Phi(x') = \begin{cases} \Phi(x) & t > t' \\ 0 & t < t' \end{cases}$$

b) $$i\int d^3x\, \bar{\Phi}(t,x)\,\gamma^0\, K(x-x') = \begin{cases} \bar{\Phi}(t',x') & t' < t \\ 0 & t' > t \end{cases}$$

(**)

$\Rightarrow$ plane wave with positive p^0 is only propagated in future

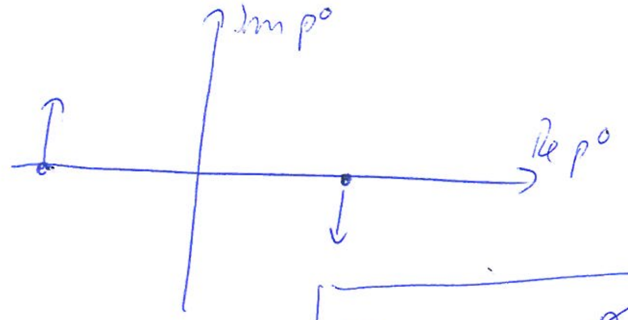
adjoint " " " " " past

(in analogy for plane waves ~~in~~ with negative p^0)

One can also avoid poles through:

$$(p^0 - E)(p^0 + E) \longrightarrow \left[p^0 + \left(E - \frac{i\varepsilon}{2E}\right)\right]\left[p^0 - \left(E - \frac{i\varepsilon}{2E}\right)\right]$$

$$= p^2 - m^2 + i\varepsilon \qquad ; \ \varepsilon > 0$$



$$\Rightarrow \boxed{\tilde{K}(p) = \frac{\not p + m}{p^2 - m^2 + i\varepsilon}}$$

ε often not written...

(note that $\frac{1}{p^2 - m^2 + i\varepsilon} = \frac{1}{2p^0}\left[\frac{1}{p^0 - \sqrt{\,} + i\varepsilon} + \frac{1}{p^0 + \sqrt{\,} - i\varepsilon}\right]$

The photon propagator: $\Box A^\mu(x) = e\, j^\mu(x)$

$$= e\, \bar{\psi}_f \gamma^\mu \psi_i(x)$$

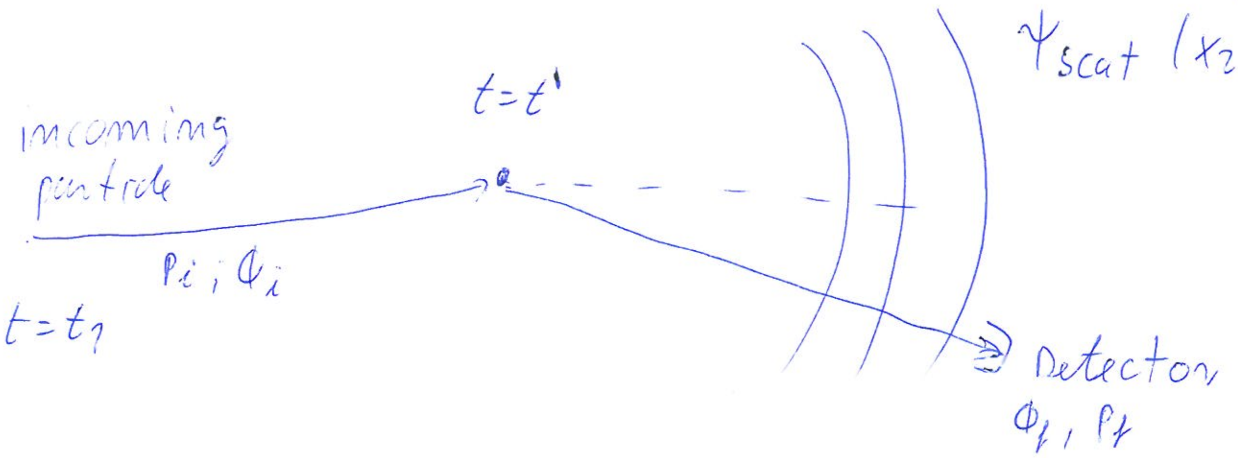
is solved by $A^\mu(x) = e \int d^4x'\, D^{\mu\nu}(x - x')\, j_\nu(x')$ $\quad$ (~~**~~ *)

if $\Box D^{\mu\nu}(x - x') = g^{\mu\nu}\, \delta(x - x')$

$$\Rightarrow \tilde{D}^{\mu\nu}(p) = -\frac{g^{\mu\nu}}{p^2 + i\varepsilon}$$

III 3) Feynman rules and elementary processes

fermions interact with A_μ; scattering



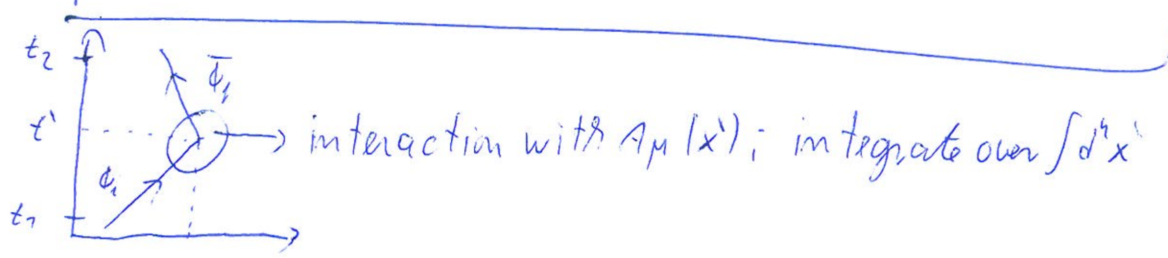
$$\Psi_{scat} = S\,\Phi_i \qquad \text{with} \qquad \Psi^{(1)}_{scat} = \Phi_i(x_2) - e\int d^4x'\, K(x_2 - x')\, \not{A}(x')\,\Phi_i(x')$$

plane wave passing through unperturbed

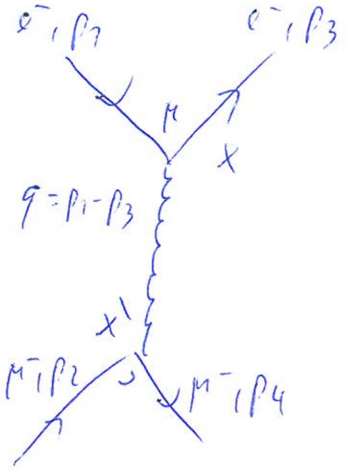
$$\Rightarrow S_{fi} = \int d^3x_2\, \Phi_f^{+}(x_2)\, \underbrace{S\,\Phi_i(x_2)}_{\Psi_{scat}} = \int d^3x_2\, \Phi_f^{+}(x_2)\, \Psi_{scat}(x_2)$$

$$= -e \int \underbrace{d^3x_2}\, d^4x'\, \underbrace{\Phi_f^{+}(x_2)\, K(x_2 - x')}_{= -i\,\overline{\Phi}_f(x') \text{ from } (**)}\, \not{A}(x')\,\Phi_i(x')$$

$$\Rightarrow \boxed{S^{(1)}_{fi} = i\,e\int d^4x'\, \overline{\Phi}_f(x')\, \not{A}(x')\, \Phi_i(x')}$$



now glue together for $e^- \mu^- \to e^- \mu^-$



we read from (***):

$$A_\mu(x) = e\int d^4x' \frac{d^4q}{(2\pi)^4} \frac{-g_{\mu\nu}}{q^2+i\varepsilon} J^\nu(x')\, e^{-iq(x-x')}$$

with $J_\nu(x') = \sqrt{\frac{1}{(2\pi)^3 2E_4}} \sqrt{\frac{1}{(2\pi)^3 2E_3}}\, \bar{u}_4 \gamma_\nu u_2\, e^{i(p_4-p_2)x'}$

↳ insert in $S_{fi}^{(1)}$:

$$\Rightarrow S_{fi} = -ie^2 \int d^4x\, d^4x' \frac{d^4q}{(2\pi)^4}\, \bar{u}_3 \gamma_\mu u_1 \frac{e^{-iq(x-x')}}{q^2+i\varepsilon}\, \bar{u}_4 \gamma^\mu u_2\, e^{i(p_3-p_1)x} e^{i(p_4-p_2)x'}$$

$$\times \sqrt{}\sqrt{}\sqrt{}\sqrt{}$$

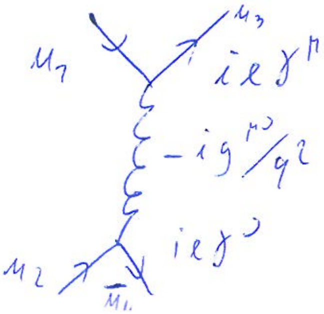
$$= \dots = +ie^2 \sqrt{}\sqrt{}\sqrt{}\sqrt{}\,(2\pi)^4\, \bar{u}_3 \gamma^\mu u_1 \frac{-g_{\mu\nu}}{q^2+i\varepsilon}\, \bar{u}_4 \gamma^\nu u_2 \;\times$$

$$\delta(p_1+p_2-p_3-p_4)$$

now recall that $S_{fi} = -i(2\pi)^4 \delta(P_f - P_i)\, T_{fi}$

and $T_{fi} = \prod_{i,f} \sqrt{\frac{1}{2E_i}} \sqrt{\frac{1}{2E_f}}\, \mathcal{M}_{fi}$

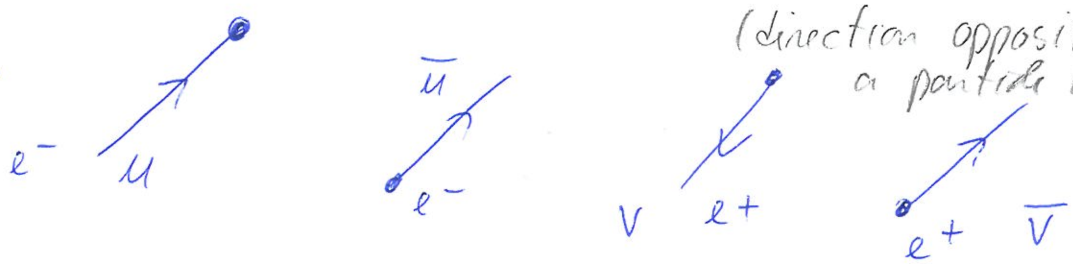
$\Rightarrow \boxed{\text{Feynman-rules for } \mathcal{M}}$



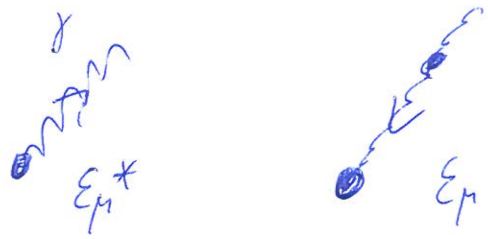
Feynman - rules:

[these are Feynman rules for $\mathcal{M}$]

- overall factor i; $E-\vec{p}$ conservation at each vertex

- 

(direction opposite to what a particle would have!)

- 

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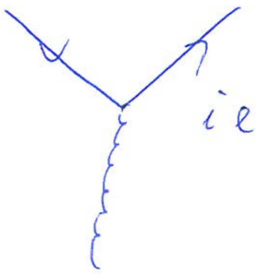
$$\frac{i}{\not p - m}$$

- 

$$\frac{-ig^{\mu\nu}}{q^2}$$

-  $ie\gamma_\mu$

„extra-rules"...

- -1 for closed fermion loop
- $\int \frac{d^4q}{(2\pi)^4}$ for closed loops
- (-1) relative factor for graphs that differ by interchange of 2 identical fermion lines
- Symmetry factor S;
  $\leftrightarrow$ # of times one can permute internal lines and vertices; with external lines fixed

$\hookrightarrow$ for cross section we need $|\mathcal{M}|^2$

$$-i\mathcal{M} = ie^2 \, \bar{u}_3 \gamma^\mu u_1 \, \bar{u}_4 \gamma_\mu u_2 / q^2$$

$\Rightarrow \overline{|\mathcal{M}|^2}$ average over initial spins/polarization and sum over final states

$$\overline{|\mathcal{M}|^2} \cdot q^4 = \frac{e^4}{2 \cdot 2} \sum_{s_f, s_i} \left[\bar{u}_3 \gamma^\mu u_1 \bar{u}_4 \gamma_\mu u_2\right]\left[u_2^\dagger \gamma_\varrho^\dagger \bar{u}_4^\dagger u_1^\dagger \gamma^{\varrho\dagger} \bar{u}_3^\dagger\right]$$

$2 \rightarrow \frac{1}{2S_A+1}$, $\quad 2 \rightarrow \frac{1}{2S_B+1}$

use that $\bar{u}^\dagger = (u^\dagger \gamma_0)^\dagger = \gamma_0^\dagger u = \gamma_0 u$

$$= \frac{e^4}{4} \sum_{s_f, s_i} \bar{u}_3 \gamma^\mu u_1 \bar{u}_4 \gamma_\mu u_2 \; \bar{u}_2 \underbrace{\gamma_0 \gamma_\varrho^\dagger \gamma_0}_{\gamma_\varrho} u_4 \; \bar{u}_1 \underbrace{\gamma_0 \gamma^{\varrho\dagger} \gamma_0}_{\gamma^\varrho} u_3$$

$$= \frac{e^4}{4} \sum_{s_f, s_i} \bar{u}_3 \gamma^\mu u_1 \bar{u}_4 \gamma_\mu \underbrace{u_2 \bar{u}_2}_{\not{p}_2 + m_2} \gamma_\varrho u_4 \bar{u}_1 \gamma^\varrho u_3$$

$$= \frac{e^4}{4} \sum (\bar{u}_3)_\alpha (\gamma^\mu)_{\alpha\beta} (u_1)_\beta (\bar{u}_4)_\gamma (\gamma_\mu)_{\gamma\delta} (\not{p}_2 + m_2)_{\delta\varepsilon} (\gamma_\varrho)_{\varepsilon\eta} (u_4)_\eta (\bar{u}_1)_\sigma (\gamma^\varrho)_{\sigma\tau} (u_3)_\tau$$

$$= \frac{e^4}{4} \operatorname{Tr}\left\{\gamma^\mu (\not{p}_1 + m_1) \gamma^\varrho (\not{p}_3 + m_3)\right\} \operatorname{Tr}\left\{\gamma_\mu (\not{p}_2 + m_2) \gamma_\varrho (\not{p}_4 + m_4)\right\}$$

$$= f(s, t, u, m_i^2)$$

$\Rightarrow$ Trace theorems:

$$\mathrm{Tr}\, \mathbb{1} = 4$$

$$\mathrm{Tr}\, \gamma_\mu = 0 = \mathrm{Tr}\, \gamma_5 \quad \text{where } \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\{\gamma_5, \gamma_\mu\} = 0 \,, \quad \gamma_5^2 = \mathbb{1} \,, \quad \gamma_5^\dagger = \gamma_5$$

$$\mathrm{Tr}\, \gamma^\mu \gamma^\nu = 4 g^{\mu\nu} \Rightarrow \mathrm{Tr}\{\not{a}\not{b}\} = 4\, a \cdot b$$

$$\mathrm{Tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} = 4\left(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}\right)$$

$\vdots$

$$\Rightarrow q^4 |\overline{M}|^2 = e^4 \cdot 4\left[2(p_1\cdot p_2)(p_3\cdot p_4) + 2(p_1\cdot p_4)(p_2\cdot p_3) + q^2\left(q^2 + (p_1\cdot p_3) + (p_2\cdot p_4)\right)\right]$$

$$\xrightarrow{m_i = 0} \; = 2 e^4 (s^2 + u^2)$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega}(e^- \mu^- \to e^- \mu^-) = \frac{\alpha^2}{2} \frac{s^2 + u^2}{s\, t^2}} \qquad \alpha = \frac{e^2}{4\pi}$$

Apply crossing: $e^+e^- \to \mu^+\mu^-$ $\Rightarrow$ $s \leftrightarrow t$ in $|\bar{\mathcal{M}}|^2$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega}(e^+e^- \to \mu^+\mu^-) = \frac{\alpha^2}{2}\,\frac{t^2+u^2}{s^3}}$$

use $t = (p_1-p_3)^2 = -2\,p_1\cdot p_3 = -2\,(E_1E_3 - |\vec{p}_1|\,|\vec{p}_3|\cos\theta)$

$$= -\frac{s}{2}(1-\cos\theta)$$

$$u = -\frac{s}{2}(1+\cos\theta)$$

$$\Rightarrow\ t^2+u^2 = \frac{s^2}{2}(1+\cos^2\theta)$$

and $\dfrac{d\sigma}{d\Omega}(e^+e^- \to \mu^+\mu^-) = \dfrac{\alpha^2}{s\cdot 4}(1+\cos^2\theta)$

and $\sigma = \dfrac{4\pi\alpha^2}{3s} \simeq \dfrac{87\ \mathrm{mb}}{[s/\mathrm{GeV}]^2}$