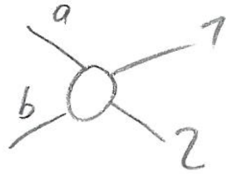
II 2) 2-to-2 cross section



$$(2\pi)^2\, d\Phi_2 = \int \frac{d^3p_1}{2E_1}\,\frac{d^3p_2}{2E_2}\,\delta(p_1+p_2-p_a-p_b)$$

$$= \int d^4p_1\, \underline{d^4p_2}\, \delta(p_1^2-m_1^2)\,\delta(\underline{p_2}^2-m_2^2)\,\Theta(E_1)\,\Theta(\underline{E_2})\,\underline{\delta(p_1+p_2-p_a-p_b)}$$

$$= \int d^4p_1\, \delta(p_1^2-m_1^2)\,\delta\left[(p_a+p_b-p_1)^2-m_2^2\right]\Theta(E_1)\,\Theta(E_a+E_b-E_1)$$

$$= \int dE_1\, \underline{d|\vec{p}_1|}\, \vec{p}_1^{\,2}\, d\Omega\, \underbrace{\delta(E_1^2-\vec{p}_1^{\,2}-m_1^2)}\ \delta[\dots]\dots$$

$$\delta(\vec{p}_1^{\,2}-(E_1^2-m_1^2)) = \frac{1}{2|\vec{p}_1|}\left[\delta\left(|\vec{p}_1|-\sqrt{E_1^2-m_1^2}\right)+\cancel{\delta(\quad +\quad)}\right]$$

due to $|\vec{p}_1|>0$

$$= \int \underline{dE_1}\, d\Omega\, \frac{|\vec{p}_1|}{2}\, \underbrace{\delta\Big[s\underbrace{-2(p_a+p_b)p_1}_{-2\sqrt{s}\,E_1 \text{ in CMS}}+m_1^2-m_2^2\Big]}_{\frac{1}{2\sqrt{s}}\delta\left[E_1-\frac{1}{2\sqrt{s}}(s+m_1^2-m_2^2)\right]}\Theta(E_1)\,\Theta(\underbrace{E_a+E_b}_{\sqrt{s} \text{ in CMS}}-E_1)$$

$$= \int d\Omega\, \frac{|\vec{p}_1|}{4\sqrt{s}}\, \underbrace{\Theta\Big(\frac{1}{2\sqrt{s}}\underbrace{(s+m_1^2-m_2^2)}_{>0}\Big)}_{=1}\, \underbrace{\Theta\Big(\underbrace{\sqrt{s}-\frac{1}{2\sqrt{s}}(s+m_1^2-m_2^2)}_{\frac{1}{2\sqrt{s}}(2s-s+m_1^2-m_2^2)>0}\Big)}_{=1}$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\sqrt{\frac{\lambda(s, m_1^2, m_2^2)}{\lambda(s, m_a^2, m_b^2)}}\, |M_{fi}|^2}$$

$$\Downarrow$$

$$\boxed{\frac{d\sigma}{dt} = \frac{d\Omega}{dt}\frac{d\sigma}{d\Omega} = \frac{1}{16\pi\, \lambda(s, m_a^2, m_b^2)}\, |M_{fi}|^2}$$

also: $\boxed{\frac{d\Gamma}{d\Omega} = \frac{1}{64\pi^2}\frac{\sqrt{\lambda(m_a^2, m_1^2, m_2^2)}}{m_a^3}|M_{fi}|^2}$ for $a \to 1+2$

Remarks: 1) typically: $|M_{fi}|^2 \to |\overline{M}_{fi}|^2$

averaged over initial spins, summed over final spins. (or polarizations)

2) n identical particles in final state:

multiply Γ, σ with $\frac{1}{n!}$ (or $\theta_1 < \theta_2 < \ldots \theta_n$)

III] QFT, QED, Feynman rules

III 1) How to describe fermions and photons

Dirac equation for free particles

$$\boxed{(i\not{\partial} - m)\psi = (\not{p} - m)\psi = 0} \quad (1)$$

$$\psi = \begin{pmatrix} \cdot \\ \vdots \end{pmatrix}, \quad \not{p} = p_\mu \gamma^\mu \quad ; \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

adjoint spinor solves $i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0$ (2)

with $\bar{\psi} = \psi^\dagger \gamma_0 = (\cdot^*, \cdot^*, -\cdot^*, -\cdot^*)$

conserved current: $\partial_\mu j^\mu = 0$ with $j^\mu = \bar{\psi}\gamma^\mu\psi$

$\bar{\psi}\cdot(1) + (2)\psi$

$\exists$ 4 solutions: $\psi_{1,2} = e^{-ipx} u_{1,2}$ $E > 0$ (e^-)

$\psi_{3,4} = e^{ipx} v_{2,1}$ $E < 0$ (e^+)

$$(\not{p} - m)u = 0 = \bar{u}(\not{p} - m)$$

$$(\not{p} + m)v = \bar{v}(\not{p} + m)$$

$$u_s = \sqrt{E+m}\begin{pmatrix}\chi_s \\ \frac{\vec{\sigma}\vec{p}}{E+m}\chi_s\end{pmatrix} \qquad \begin{aligned}\chi_1 = \chi_{1/2} &= \begin{pmatrix}1\\0\end{pmatrix} \text{ (up) spin } \parallel \vec{p} \\ \chi_2 = \chi_{-1/2} &= \begin{pmatrix}0\\1\end{pmatrix} \text{ (down) spin anti} \parallel \vec{p}\end{aligned}$$

$$v_s = \sqrt{E+m}\begin{pmatrix}\frac{\vec{\sigma}\vec{p}}{|E|+m}\phi_s \\ \phi_s\end{pmatrix} \qquad \begin{aligned}\phi_1 &= \begin{pmatrix}0\\1\end{pmatrix} \quad \text{Spin up} \\ \phi_2 &= \begin{pmatrix}1\\0\end{pmatrix} \quad \text{Spin down}\end{aligned}$$

Normalization:

$$\boxed{\begin{aligned} &\bar{u}_s u_{s'} = 2m\,\delta_{ss'} = -\bar{v}_s v_{s'} \\ &\sum_s u_s \bar{u}_s = \not{p} + m \\ &\sum_s v_s \bar{v}_s = \not{p} - m \\ &u_s^+ u_{s'} = v_s^+ v_{s'} = 2E\,\delta_{ss'}\end{aligned}}$$

most general solution:

$$\overset{(-)}{\psi}(x) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3 2E}}\left[a_s^{(+)}(p)\,\overset{(-)}{u}_s(p)\,e^{\mp ipx} + b_s^{+()}(p)\,\overset{(-)}{v}_s(p)\,e^{\pm ipx}\right]$$

2nd quantization (also: canonical):

$$\begin{aligned} a &\to \hat{a} \\ b^+ &\to \hat{b}^+ \end{aligned} \qquad \psi \to \psi^+ \qquad \underline{\text{Operators}}$$

$\hat{a}$: annihilates particle ; $\hat{b}^+$: creates antiparticle

Interaction with photon field

"minimal substitution" $p^\mu \to p^\mu - qA^\mu$ $\quad (p = i\partial)$

"covariant derivative" $\partial^\mu \to \partial^\mu + iqA^\mu$

here: $q(e^-) = -e \;;\; e > 0$

$$\Rightarrow \left(i\not{\partial} - q\not{A} - m \right) \psi = 0$$

or, in Lagrangian

$$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi - q\bar{\psi}\gamma^\mu A_\mu \psi}$$

field strength tensor

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

free particle

$\mathcal{L}_0$

interaction term

$\mathcal{L}_I$

Apply Euler-Lagrange: $\partial_\mu F^{\mu\nu} = q\, j^\nu$ and $\partial_\nu j^\nu = 0$

$$\boxed{A_\mu = \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} \sum_{\lambda=0}^{3} \varepsilon_\mu(k,\lambda) \left[a(k,\lambda) e^{-ikx} + a^\dagger(k,\lambda) e^{ikx} \right]}$$

with $\sum_{\lambda=1}^{2} \varepsilon_\mu^* \varepsilon_\nu = -g_{\mu\nu}$ for real photons

(only 2 pol. degrees $\leftrightarrow$ Sheet 3)

III 2) How to propagate fermions and photons

$$(i\not\partial - m)\psi(x) = -e\not{A}\psi \qquad (*)$$

how to solve ~~/~~ handle?

Reminder: electrodynamics: $\vec{\nabla}^2 \phi(\vec{x}) = -\rho(\vec{x})$ [Poisson]

ϕ: potential, ρ: charge density

for point charge: $\rho(\vec{x}) = q\,\delta(\vec{x}-\vec{x}')$: $\phi(x) = \dfrac{q}{4\pi|\vec{x}-\vec{x}'|}$

continuous charge distribution: $\phi(x) = \displaystyle\int \frac{\rho(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}\, d^3x'$
(integration over potentials of individual charges)

alternative derivation: introduce <u>Green-function</u> $G(\vec{x},\vec{x}')$

$$\vec{\nabla}^2 G(\vec{x},\vec{x}') = -\delta(\vec{x}-\vec{x}')$$

if we set $\phi(\vec{x}) = \int d^3x'\, G(\vec{x},\vec{x}')\,\rho(\vec{x}')$, then Poisson is solved:

$$\vec{\nabla}^2\phi = \int d^3x'\, \vec{\nabla}^2 G(\vec{x},\vec{x}')\,\rho(\vec{x}') = -\rho(\vec{x})$$

$$\Rightarrow G(\vec{x},\vec{x}') = G(\vec{x}-\vec{x}') = \frac{1}{4\pi|\vec{x}-\vec{x}'|}$$

$\Rightarrow$ same approach for (*)

$$(i\not\partial - m)\, K(x, x') = \delta(x - x')$$

if K is known, then $\boxed{\psi(x) = -e\int d^4x'\, K(x-x')\, \not A(x')\,\psi(x')}$ (**)

solves (*), because:

$$(i\not\partial - m)\,\psi(x) = -e\int d^4x'\, \underbrace{(i\not\partial - m)\, K(x-x')}_{\delta(x-x')}\, \not A(x')\,\psi(x')$$

$$= -e\,\not A(x)\,\psi(x)$$

(**) is integral equation for ψ, solve iteratively

$\longleftrightarrow \alpha = \frac{e^2}{4\pi} \ll 1$

to solve, note that to RHS of (*) a free particle solution (plane wave) $\Phi(x)$ can be added: (on $\operatorname{sim}(i\not\partial - m)\psi(x)$ the free solution Φ drops out)

$$\Rightarrow \psi(x) = \Phi(x) - e\int d^4x'\, K(x-x')\, \not A(x')\,\psi(x') \qquad (***)$$

first approximation: $\psi^{(0)}(x) = \Phi(x)$; insert in (***)

$$\Rightarrow \psi^{(1)}(x) = \Phi(x) - e\int d^4x'\, K(x-x')\,\not A(x')\,\Phi(x')$$

insert in (***)

$$\Rightarrow \psi^{(2)}(x) = \Phi(x) - e\int d^4x'\, K(x-x')\,\not A(x')\,\Phi(x') + e^2\int d^4x'\, d^4x''\, K(x-x'')\,\not A(x'')\,K(x''-x')\,\times$$
$$\not A(x')\,\Phi(x')$$