Recap: QED-Lagrangian, existence of massless photon field, from invariance under local Abelian $U(1)$-Symmetry:

$$\partial_\mu \rightarrow D_\mu = \partial_\mu - ie A_\mu$$

$$A_\mu \rightarrow A_\mu' = A_\mu + \frac{1}{e}(\partial_\mu \alpha) \; ; \; F_{\mu\nu} = \frac{i}{e}[D_\mu, D_\nu]$$

$$\psi \rightarrow \psi' = e^{i\alpha(x)}\psi$$

$$\Rightarrow D_\mu' \psi' = e^{i\alpha} D_\mu \psi$$

we need to generalize this principle to Non-Abelian $SU(N)$-theories $SU(N)$:

$$U = \exp\{-i\Theta^a t^a\} \simeq 1 - i\Theta^a t^a \equiv 1 - i\vec{\Theta}\vec{t}$$

$$a = 1, N^2-1$$

As U is $N \times N$ matrix, multiply it with an N-component object $\psi_i = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix}$

Formalism of Non-Abelian gauge symmetries

$$\boxed{\psi_i \to \psi_i' = U_{ij}\,\psi_j}$$ with $U_{ij} = \exp\{-i\theta^a t^a\}_{ij}$

$= (1 - i\theta^a t^a)_{ij}$

invariance of $\bar\psi(\not{\partial} - m)\psi$ is achieved through:

$$\boxed{\begin{aligned} \partial_\mu \to D_\mu &= \partial_\mu - ig A_\mu^a t^a \equiv \partial_\mu - ig \vec{A}\vec{t} \equiv \partial_\mu - ig \tilde{A}_\mu \\ \tilde{A}_\mu \to \tilde{A}_\mu' &= \frac{-i}{g}(\partial_\mu U)U^{-1} + U\tilde{A}_\mu U^{-1} \end{aligned}}$$

such that $D_\mu'\psi' = U D_\mu \psi$ or $\not{D}' = U\not{D}U^{-1}$

<u>proof:</u> $\underline{D_\mu'\psi'} = \left[\partial_\mu - ig\left(\frac{-i}{g}(\partial_\mu U)U^{-1} + U\tilde{A}_\mu U^{-1}\right)\right]U\psi$

$= \cancel{(\partial_\mu U)\psi} + U(\partial_\mu\psi) - \cancel{(\partial_\mu U)\psi} - igU\tilde{A}_\mu\psi$

$= U(\partial_\mu - ig\tilde{A}_\mu)\psi = \underline{U D_\mu\psi}$

if this holds, then obviously

$$\bar\psi'(\not{D}' - m)\psi' = \bar\psi' U^{-1}(U D_\mu - m)\psi = \bar\psi(\not{D} - m)\psi$$

•) infinitesimal trafo:

$$A_\mu^a t^a \to (1 - i\Theta^b t^b) A_\mu^a t^a (1 + \Theta^b t^b)$$

$$- \frac{i}{g} \left[\partial_\mu (1 - i\Theta^a t^a) \right] (1 + i\Theta^b t^b)$$

$$= A_\mu^a t^a + i\Theta^b A_\mu^a \underbrace{[t^a, t^b]}_{if^{abc} t^c} - \frac{1}{g} (\partial_\mu \Theta^a) t^a$$

$$= A_\mu^a t^a - \frac{1}{g} (\partial_\mu \Theta^a) t^a + \underbrace{f^{abc} \Theta^b A_\mu^c t^a}_{\text{new term!}}$$

•) kinetic terms

$$F_{\mu\nu} = \frac{i}{g} [D_\mu, D_\nu] = F_{\mu\nu}^a t^a$$

$$= \dots = \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \underbrace{g f_{abc} A_\mu^b A_\nu^c}_{\text{new term!}} \right) t^a$$

trafo: $[D'_\mu, D'_\nu] = [U D_\mu U^{-1}, U D_\nu U^{-1}] = U [D_\mu, D_\nu] U^{-1}$

__not__ gauge invariant...

BUT: $\mathrm{Tr}\{F_{\mu\nu} F^{\mu\nu}\}$ is gauge invariant!

Note: •) massive gauge bosons $m_W^2 A_\mu^a A^{\mu,a}$

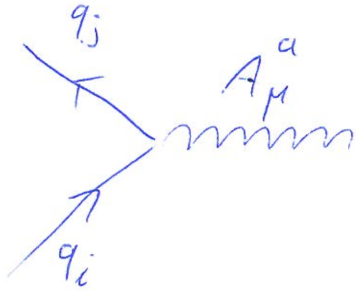
still forbidden

•) A_μ^a : N^2-1 gauge bosons!

•) $F_{\mu\nu} F^{\mu\nu} \sim (\partial A + g A^2)^2 \sim g A^3 + g^2 A^4$

self-interactions!!

•)



$(t^a)_{ij}$

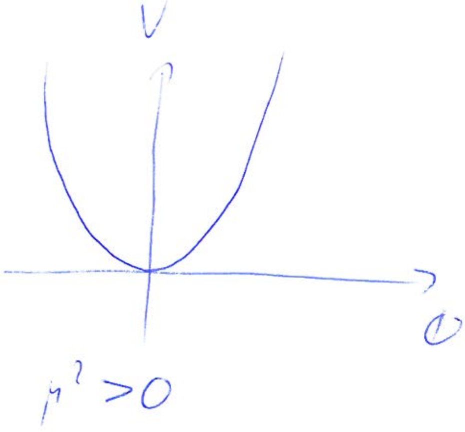
from $\mathcal{L} \subset ig \bar{\psi} \gamma^\mu t^a \psi A_\mu^a$

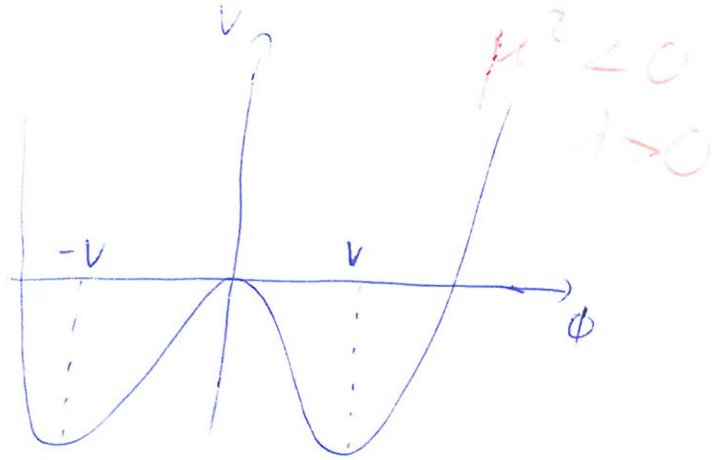
3) Spontaneous Symmetry Breaking and the Higgs-mechanism!

$\mathcal{L}$ possesses symmetry which the ground state does not obey

a) $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi) \; ; \; V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi$

Symmetry: $\phi \to -\phi$ („ Z_2 ")





$$\boxed{\frac{\partial V}{\partial \phi^2} = 0 \Rightarrow v = \pm\sqrt{-\mu^2/\lambda}}$$

VACUUM EXPECTATION VALUE

choose $\langle \phi \rangle = +v$ and do physics around the minimum $\quad \phi = v + \eta(x)$

(we can only do perturbation theory)

$$\Rightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \lambda v^2 \eta^2 - \lambda v \eta^3 - \frac{1}{2}\lambda \eta^4$$

$$\Rightarrow \boxed{m_\eta^2 = 2\lambda v^2}$$

correct sign for mass term

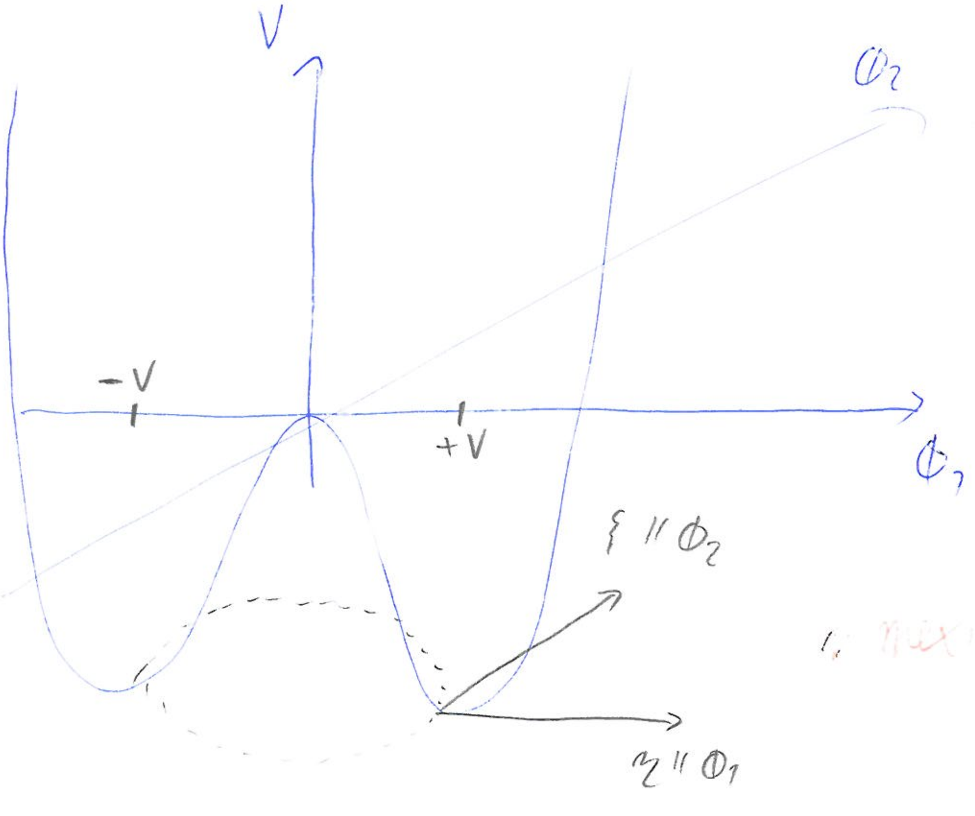
generated by <u>spontaneous symmetry breaking</u>.

The $\Phi \to -\Phi$ symmetry is no longer present in the ground state.

b) $\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi^*) - \mu^2 \Phi^* \Phi - \lambda (\Phi^* \Phi)^2$

with $\Phi = \sqrt{\frac{1}{2}}(\Phi_1 + i\Phi_2)$ complex

Symmetry : $\Phi \to e^{i\alpha}\Phi$ global $U(1)$



choose

$\langle \Phi \rangle = v = \sqrt{-\mu^2/\lambda}$

with

$\Phi = \sqrt{\frac{1}{2}}(v + \eta + i\xi)$

$\eta(x),\ \xi(x)$

„mexican hat"

$\Rightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu \xi)^2 + \frac{1}{2}(\partial_\mu \eta)^2 \quad + \mu^2 \eta^2 + (\text{cubic, quartic}$

$\Rightarrow$ massive η

massless ξ "flat direction", feels no resistance when moving

"Goldstone - Boson" ξ, occurs always when a continuous symmetry is broken spontaneously

c) $\mathcal{L}_0 = \left[(\partial^\mu + ie A^\mu)\phi^*\right]\left[(\partial_\mu - ie A_\mu)\phi\right] - \mu^2 \phi^* \phi - \lambda(\phi^* \phi)^2$

Symmetry: $\phi \to e^{i\alpha(x)}\phi(x)$ local $U(1)$

(recall: $A_\mu \to A_\mu + \frac{1}{e}(\partial_\mu \alpha)$)

choose : $\phi = \sqrt{\frac{1}{2}}\,(v + \eta(x) + i\xi(x))$

$$\Rightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu \xi)^2 + \frac{1}{2}(\partial_\mu \eta)^2 - v^2 \lambda \eta^2 + \frac{1}{2} e^2 v^2 A_\mu A^\mu$$

$$- e v A_\mu \partial^\mu \xi \quad + \dots$$

$$\Rightarrow \boxed{\begin{array}{l} m_\eta^2 = 2 v^2 \lambda \\ \boxed{m_\gamma^2 = \cancel{1}\, e^2 v^2} \end{array}}$$

massive gauge boson!

what about $A_\mu \partial^\mu \xi$ term?

look more closely: before : $(\phi$ + massless $A_\mu)$ $2+2$
at d.o.f.

after : $(\xi, \eta,$ massive $A_\mu)$ $2+3$

WTF?

Note that : $\phi = \sqrt{\frac{1}{2}}\,(v + \eta + i\xi) \simeq \sqrt{\frac{1}{2}}\,(v + \eta)\, e^{i \xi / v}$

this looks like a gauge trafo!

$\Rightarrow$ make a gauge : ("unitary gauge") such that η is reel

$$\boxed{\begin{aligned} &\Phi \to \Phi' = (v+\eta)/\sqrt{2}\, e^{i\xi/v} \\ &A_\mu \to A'_\mu = A_\mu + \frac{1}{ev}(\partial_\mu \xi) \end{aligned}}$$

and insert in original $\mathcal{L}_0$

$$\Rightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu \eta)^2 - \lambda v^2 \eta^2 + \frac{1}{2} e^2 v^2 A_\mu^2 + \dots$$

$\Rightarrow$ ξ disappears from $\mathcal{L}$!

$\Rightarrow$ degrees of freedom: η + massive A_μ : $1+3=4$

ξ has been „eaten" by A_μ, becoming its third degree of freedom

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consider $(\partial^\mu + ieA^\mu + \frac{i}{v}(\partial_\mu \xi))(v+\eta)e^{-i\xi/v}$

$$= (\partial^\mu \eta)e^{-i\xi/v} + (v+\eta)\left(\frac{-i\partial_\mu \xi}{v}\right)e^{-i\xi/v} + ieA^\mu(v+\eta)e^{-i\xi/v} + \left(\frac{i}{v}\partial_\mu \xi\right)(v+\eta)e^{-i\xi/v}$$

$$= (\partial^\mu \eta)e^{-i\xi/v} + ieA^\mu(v+\eta)e^{-i\xi/v}$$

$$\Rightarrow |\;|^2 = (\partial^\mu \eta)^2 + e^2v^2 A_\mu A^\mu + e^2 A_\mu A^\mu \eta^2 + 2e^2 v A_\mu A_\mu \eta$$