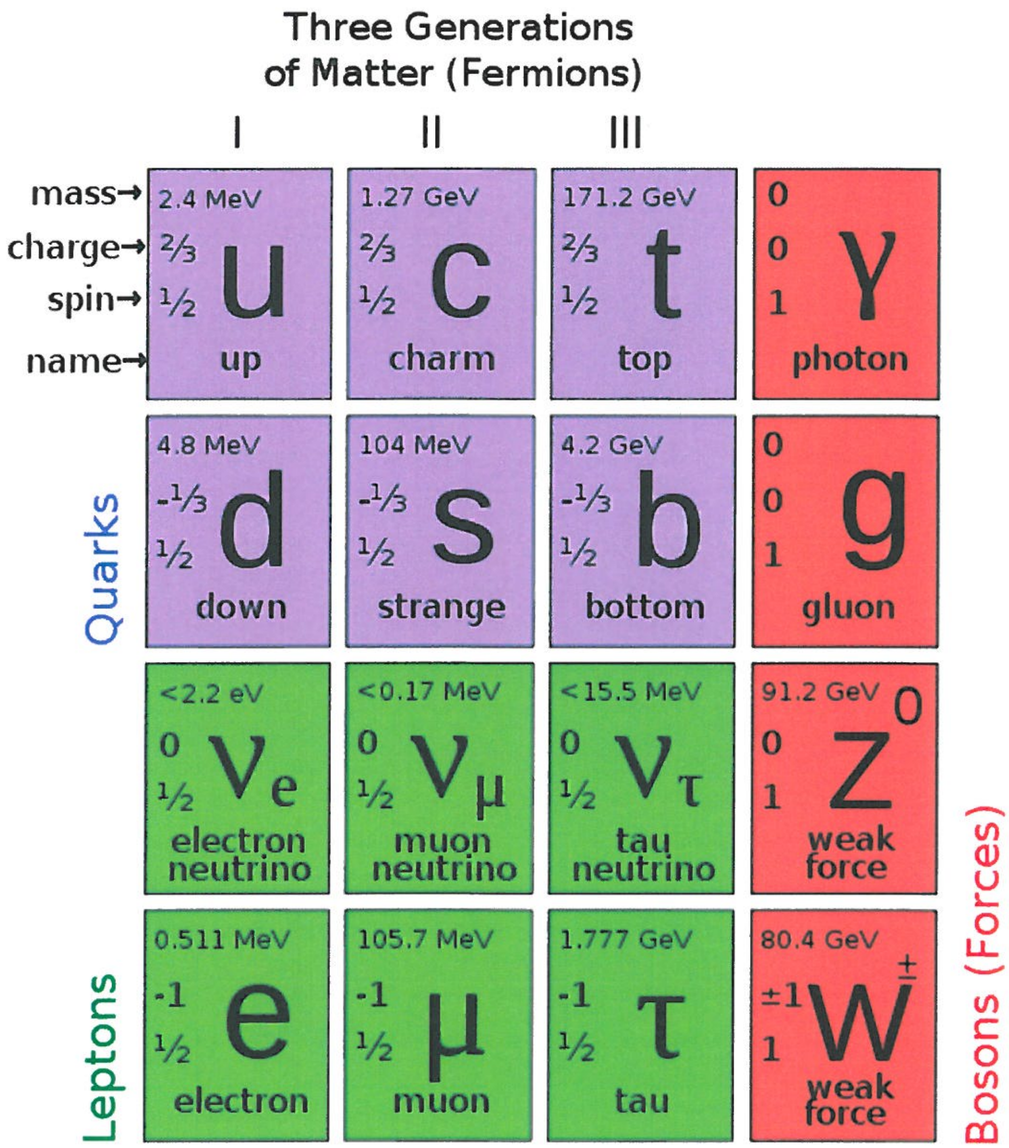
Gauge Theories and the Standard Model

3 lectures of theory behind the SM

- Gauge symmetries: Abelian and Non-Abelian

- $SU(3) \times \underbrace{SU(2) \times U(1)}_{\text{focus...}}$

- Electroweak symmetry breaking and the Higgs



(+ Higgs-boson)

- SM will describe interactions + the fact that some are massive, others are not massive

- <u>cannot</u> tell us: why I, II, III? why chiral? parameter values? Fermion mixing? Dark matter? Baryon asymmetry? Conservation of L and B? ....

1) The gauge principle

fermions; bosons; scalars

$$(\not p - m)\psi = 0 \; ; \; \left[\partial_\mu \partial^\mu (+m^2)\right] A^\nu = 0 \; ; \; (\partial_\mu \partial^\mu + m^2)\phi = 0$$

described by Lagrange density (Lagrangian) $\mathcal{L}$

Action: $S = \int d^4x \, \mathcal{L}$ $\qquad [S] = [\hbar]$

principle of ~~last~~ least action: $\boxed{\delta S = 0}$

$$\Rightarrow \boxed{\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi}} \qquad (\text{or } \psi \text{ or } A^\nu)$$

Euler Lagrange equation

$$\mathcal{L} = \bar\psi (\not p - m)\psi \qquad (*)$$

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \phi)(\partial^\mu \phi) - m^2 \phi^2\right]$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\nu A_\mu + \frac{1}{2} m^2 A^\mu A_\mu$$

We note that (*) is invariant under

$$\psi \to e^{i\alpha}\psi \simeq (1+i\alpha)\psi \equiv \psi'$$

<u>global</u> gauge transformation

$\alpha = const$

insert in $\mathcal{L}$ and use that $\delta\mathcal{L} = 0$:

$$\Rightarrow 0 = \frac{\partial \mathcal{L}}{\partial \psi}\delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\delta(\partial_\mu \psi)$$

$$= \frac{\partial \mathcal{L}}{\partial \psi}\underbrace{(\psi' - \psi)}_{i\alpha\psi} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\underbrace{\left[\partial_\mu \psi' - \partial_\mu \psi\right]}_{i\alpha\,\partial_\mu \psi}$$

$$= i\alpha\underbrace{\left[\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right)\right]}_{=0}\psi + i\alpha\,\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\psi\right)$$

$$\Rightarrow i\alpha\,\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\psi\right) = 0$$

with $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \frac{\partial}{\partial(\partial_\mu \psi)}\left[\bar\psi(i\gamma_\mu \partial^\mu - m)\psi\right] = i\bar\psi\gamma^\mu\psi$;

$$\boxed{\partial_\mu j^\mu = 0 \quad \text{with} \quad j^\mu = \bar\psi\gamma^\mu\psi}$$

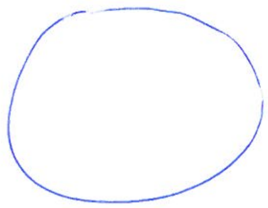
$\Rightarrow$ conserved current due to invariance under global trafo

„Abelian $U(1)$ gauge transformation"

„Noether theorem"

also leads to conserved charge:

$$\frac{d}{dt}Q = \frac{d}{dt}\int d^3x\, j^0 = -\int d^3x\, \vec\nabla\vec j \underset{\text{Gauß}}{=} -\oint d\vec S\cdot\vec j = 0$$



Volume V

$\psi(x=\infty)$ @ surface $\to 0$

charge constant if no source leaves volume, or: any source leaving V must be accounted for by flux leaving

Generalization: $\alpha \to \alpha(x)$ local „gauge" transformation

problem: $\bar{\Psi}(i\not{\partial} - m)\Psi \to \bar{\Psi}\, e^{-i\alpha(x)}(i\not{\partial} - m)\, e^{i\alpha(x)}\Psi$

$$= -\bar{\Psi} m \Psi + \bar{\Psi} e^{-i\alpha}\left[i(\not{\partial}\Psi)e^{i\alpha} \cancel{\bar{}}\, i(\not{\partial}\alpha)\, e^{i\alpha}\Psi\right]$$

$$= \bar{\Psi}(i\not{\partial} - m)\Psi - \bar{\Psi}\gamma^{\mu}\Psi(\partial_{\mu}\alpha)$$

solution: $\mathcal{L} = \bar{\Psi}(i\not{D} - m)\Psi \to \bar{\Psi} e^{-i\alpha}(i\not{D}' - m)e^{i\alpha}\Psi$

such that $D_{\mu}' = e^{i\alpha} D_{\mu}$

only possibility to achieve this:

$$\boxed{\begin{aligned} D_{\mu} &= \partial_{\mu} - ieA_{\mu} \\ A_{\mu} &\to A_{\mu} + \frac{1}{e}\partial_{\mu}\alpha \end{aligned}}$$

covariant derivative

gauge field

proof: $D_{\mu}'\Psi' = (\partial_{\mu} - ieA_{\mu} - i\partial_{\mu}\alpha)e^{i\alpha}\Psi$

$$= \underline{i(\partial_{\mu}\alpha)e^{i\alpha}\Psi} + e^{i\alpha}(\partial_{\mu}\Psi) - ieA_{\mu}e^{i\alpha}\Psi \underline{- i(\partial_{\mu}\alpha)e^{i\alpha}\Psi}$$

$$= e^{i\alpha}(\partial_{\mu} - ieA_{\mu})\Psi = e^{i\alpha} D_{\mu}\Psi$$ ☺

$\Rightarrow$ total Lagrangian:

$$\boxed{\mathcal{L}_{tot} = \bar{\Psi}(\not{p} - m)\Psi + e\bar{\Psi}\gamma^\mu\Psi A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$$

free particle — interaction with gauge field — kinetic term

Note: 1) $F'_{\mu\nu} = (\partial_\mu A'_\nu - \partial_\nu A'_\mu) = \partial_\mu(A_\nu - \tfrac{1}{e}\partial_\nu\alpha) - \partial_\nu(A_\mu - \tfrac{1}{e}\partial_\mu\alpha)$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{e}\underbrace{\left(-\partial_\nu(\partial_\mu\alpha) + \partial_\mu(\partial_\nu\alpha)\right)}_{=0}$$

$= F_{\mu\nu}$: is gauge invariant

2) $\frac{i}{e}[D_\mu, D_\nu] = \frac{i}{e}[\partial_\mu - ieA_\mu, \partial_\nu - ieA_\nu]$

$$= \frac{i}{e}(-ie)\left\{[A_\mu, \partial_\nu] + [\partial_\mu, A_\nu]\right\}$$

$$= A_\mu\partial_\nu - \partial_\nu A_\mu + \partial_\mu A_\nu - A_\nu\partial_\mu$$

$$= \underline{A_\mu\partial_\nu} - (\partial_\nu A_\mu) - \underline{A_\mu\partial_\nu} + (\partial_\mu A_\nu) + A_\nu\partial_\mu - A_\nu\partial_\mu$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$$

3) $m_\gamma^2 A_\mu A^\mu \rightarrow m_\gamma^2 (A_\mu + \frac{1}{e} \partial_\mu \alpha)(A^\mu + \frac{1}{e} \partial^\mu \alpha)$

$= m_\gamma^2 (A_\mu A^\mu) + \dots$ <u>not</u> gauge invariant

$\Rightarrow m_\gamma = 0$

<u>Lesson:</u>

local symmetry

$\downarrow$ needs

massless gauge field

$\downarrow$ is reason for

interactions

$\Rightarrow$ generalize this and apply to all interactions

2) Non-Abelian gauge symmetries

$$e^{i\alpha(x)} e^{i\beta(x)} = e^{i\beta(x)} e^{i\alpha(x)}$$ <u>Abelian</u> $U(1)$ group

Nature has chosen also <u>Non-Abelian</u> groups

$SU(N)$ the group of unitary $N \times N$ matrices with $\det = +1$

„Lie-group" every rotation can be expressed as some amount of infinitesimal rotations $\Rightarrow$ enough to consider those infinitesimal rotations

$$\boxed{U = e^{i\vec{\alpha}\vec{t}} \simeq 1 + i\vec{\alpha}\vec{t} = 1 + i\alpha^a t^a}$$

α_a: infinitesimal parameters

t_a: „generators" of the group $SU(N)$

properties: $UU^\dagger = 1 + i\alpha^a(t^a - t^{a\dagger}) \Rightarrow \boxed{t^{a\dagger} = t^a}$

$$\det U = \det e^{i\alpha^a t^a} = e^{\mathrm{Tr}\{\alpha^a t^a\}} \Rightarrow \boxed{\mathrm{Tr}\{t^a\} = 0}$$

there are $N^2 - 1$ generators, forming a Lie-Algebra:

$$[t^a, t^b] = i \underbrace{f^{abc}}_{\substack{\text{(antisymmetric)} \\ \text{structure constants}}} t^c$$

note: commutator is traceless and anti-hermitian

Examples: 1) SU(2) : 3 Generators

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\,\varepsilon_{ijk}\,\frac{\sigma_k}{2}$$

2) SU(3) : 8 Generators (Gell-Mann matrices

$$\left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right] = i\,f^{ijk}\,\frac{\lambda_k}{2}$$

- $(t^a)_{jk} = -i f_{ajk}$ is (N^2-1)-dim adjoint repr.
- there are 1, 2, 3, 4, ... dimensional representations of SU(N) (think spin 0, ½, 1, 3/2, .. in SU(2))

the N-dimensional is called fundamental repr.,

one can form all other representations from it

e.g. $2 \otimes 2 = 3 \oplus 1$ in spin SU(2)

- if we use SU(N), then we must apply it to a N-component object, e.g. $Q = \begin{pmatrix} u \\ d \end{pmatrix}$ and SU(2):

$$Q \to UQ \;;\; \bar{Q} \to \bar{Q}\,U^{+}$$

UQ is called irreducible representation