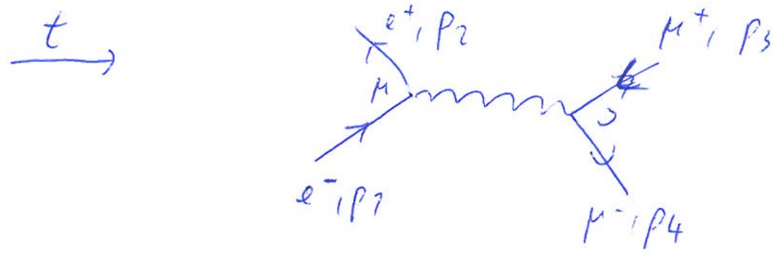
Recap. / Explanation:

Feynman-rules:

a) Antiparticles: $e^+e^- \to \mu^+\mu^-$



$$-i\mathcal{M} = \bar{v}(p_2)\, ie\gamma^\mu u(p_1) \frac{-ig_{\mu\nu}}{q^2+i\varepsilon} \overset{\bar{u}}{\cancel{v}}(p_3)\, ie\gamma^\nu \overset{v}{\cancel{u}}(p_4)$$

for $|\mathcal{M}|^2$, need $\mathcal{M}\mathcal{M}^\dagger$. Consider first term $\bar{v}_2\gamma^\mu u_1$

$$\bar{v}_2\gamma^\mu u_1 \; u_1^\dagger \gamma^{\nu\dagger}\bar{v}_2^\dagger = \bar{v}_2\gamma^\mu u_1 \; \underbrace{u_1^\dagger\gamma_0}_{\bar{u}_1}\underbrace{\gamma_0\gamma^{\nu\dagger}\gamma_0}_{\gamma^\nu} v_2$$

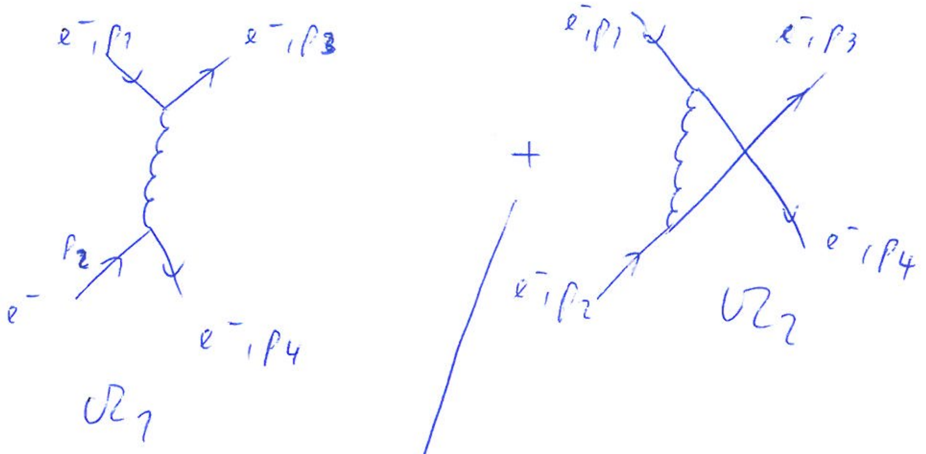
$$= \bar{v}_2\gamma^\mu u_1 \bar{u}_1 \gamma^\nu v_2$$

$$= (\bar{v}_2)_\alpha (\gamma^\mu)_{\alpha\beta} (\not{p}_1 + m_1)_{\beta\gamma} (\gamma^\nu)_{\gamma\delta} (v_2)_\delta$$

$$= (\gamma^\mu)_{\alpha\beta} (\not{p}_1 + m_1)_{\beta\gamma} (\gamma^\nu)_{\gamma\delta} \underbrace{(v_2\bar{v}_2)}_{\not{p}_2 - m_2}{}_{\delta\alpha}$$

$$= \mathrm{Tr}\left\{\gamma^\mu(\not{p}_1 + m_1)\gamma^\nu(\not{p}_2 - m_2)\right\}$$

b) $e^-e^- \to e^-e^-$



$$\mathcal{M} = \mathcal{M}_1 - \mathcal{M}_2$$

relative minus sign!
(interchange of identical fermions)
Pauli principle

$$|\mathcal{M}|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 - 2\,\mathrm{Re}\,\mathcal{M}_1^*\mathcal{M}_2$$

c) For future applications

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi - e\bar{\Psi}\gamma_\mu A^\mu \Psi$$

$\Rightarrow$ For $e\bar{\Psi}\gamma_\mu A^\mu\Psi$:

$ie\gamma_\mu$

$\Rightarrow$ read Feynman rules from Lagrangian!

Fermi-Theory

so far, QED: $e^+e^- \to \mu^+\mu^-$ etc.

also: $\pi^0 \to \gamma\gamma$ with $\tau \sim 10^{-16}$ s

BUT: $\pi^+ \to \mu^+ \nu_\mu$
$\mu \to e\, \bar{\nu}_e\, \nu_\mu$ with $\tau \sim 10^{-(8\ldots 6)}$ s

Solution: <u>weak interactions</u> Fermi-theory

1) Fundamentals

at low energies: successful descriptions

$$\mathcal{L} = G\,(\bar{f}\,\Gamma\, f')\,(\bar{f}''\,\Gamma\, f''') \qquad \left([G] = E^{-2}\right)$$

4-fermion interaction, point like 

recall: Γ could be

$\mathbb{1}$	<u>S</u>calar
γ^μ	<u>V</u>ector
$\sigma^{\mu\nu}$	<u>T</u>ensor
$\gamma^5\gamma^\mu$	<u>A</u>xial vector
γ^5	<u>P</u>seudoscalar

(or any linear combination)

Experiments show that: ~~…~~ $\gamma^\mu(1-\gamma_5)$ has been chosen by nature. $\boxed{\text{V} - \text{A Theory}}$

$$\Rightarrow \boxed{\mathcal{L} = \frac{1}{\sqrt{2}} G_F \, J_\mu^\dagger J^\mu \quad \text{with } J_\mu = \sum_{\ell,\ell'} \bar{\psi}_\ell \gamma_\mu (1-\gamma_5) \psi_{\ell'}}$$

for instance: $J^\mu = \bar{e}\,\gamma^\mu(1-\gamma_5)\nu_e$ — charge lowered by one unit

$= \bar{u}\,\gamma^\mu(1-\gamma_5)\,d$

$= \dots$

Note that $\boxed{P_{L,R} \equiv \frac{1}{2}(1 \mp \gamma_5)}$ are projection operators for CHIRALITY (HANDEDNESS)

consider up-quark (down-quark) u (d)

$$P_L u \equiv u_L \; ; \; \bar{u}_L = \bar{u} P_R$$

$$\bar{u}_L u_L = \bar{u}_R u_R = 0$$

$$\bar{u} u = \bar{u}_R u_L + \bar{u}_L u_R = \bar{u}_L u_R + h.c.$$

$$\bar{u}\gamma^\mu d_L = \frac{1}{2}\bar{u}\gamma^\mu(1-\gamma_5)d = \frac{1}{2}\bar{u}(1+\gamma_5)\gamma^\mu d = \bar{u}_L \gamma^\mu d$$

$$\bar{u}_L \gamma^\mu d_L = \bar{u} P_R \gamma^\mu P_L d = \bar{u} P_R P_R \gamma^\mu d = \bar{u} P_R \gamma^\mu d = \bar{u}_L \gamma^\mu d$$

$\Rightarrow$ V − A selects only left-handed particles!

PARITY VIOLATION!

$\psi \xrightarrow{P} \gamma_0 \psi$

$\bar{\psi}\gamma^\mu\psi \xrightarrow{P} {+\bar{\psi}\gamma^0\psi \atop -\bar{\psi}\gamma^i\psi}$ | $\bar{\psi}\gamma_\mu\gamma_5\psi \xrightarrow{P} {-\bar{\psi}\gamma_0\gamma_5\psi \atop +\bar{\psi}\gamma_i\gamma_5\psi}$ $\Big\} \Rightarrow J_V^\mu J_{V,\mu}$ ✓ $J_A^\mu J_{A,\mu}$ ✓ but not $J_A^\mu J_{V,\mu}$

recall: chirality vs. helicity

(only chirality is Lorentz-invariant

only helicity is a good quantum number)

pos. hel.   neg. hel.

helicity $\equiv$ chirality for $m = 0$:

$$\left.\begin{array}{l} P_L v_1 = v_1 \;;\; P_L v_2 = 0 \\ P_R v_1 = 0 \;;\; P_R v_2 = v_2 \\ P_L u_1 = 0 \;;\; P_L u_2 = u_2 \\ P_R u_1 = u_1 \;;\; P_R u_2 = 0 \end{array}\right\}$$

Neutrinos have ~~pos.~~ neg. hel.

Antineutrinos have pos. hel.

for $m \neq 0$, e.g. $P_L u_1 \simeq \frac{m}{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} \propto v_1(m=0)$

in general: only LH particles take part in weak interactions

$\Rightarrow$ only RH antiparticles take part in weak interactions

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$\psi^c = i\gamma_2\gamma_0\bar{\psi}^T$ is wave function of positron in Dirac-equation

$(\psi^c)_L = (\psi_R)^c$

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at this stage, CP is still conserved

Size of G_F:

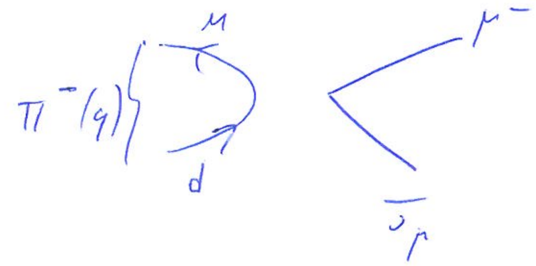
$$\mathcal{L} = \frac{4}{\sqrt{2}} G_F \left(\overline{\mu_L}\, \gamma^\mu (\nu_\mu)_L\right)\left(\overline{(\nu_e)_L}\, \gamma_\mu e_L\right)$$

$$\Rightarrow \Gamma(\mu \to \nu_\mu e^- \bar{\nu}_e) = \frac{G_F^2 m_\mu^5}{192\pi^3} \simeq 2.2 \cdot 10^{-6} s$$

$$\Rightarrow G_F = 1.166 \cdot 10^{-5} \text{ GeV}^{-2}$$
$$\simeq \left(\frac{1}{300 \text{ GeV}}\right)^2$$

2) Basic processes

a) Pion decay



$$\mathcal{M} = \frac{2G_F}{\sqrt{2}}\, \overline{\mu_L}\, \gamma_\mu \nu_L \; X^\mu$$

pionic physics

$\longrightarrow$ π^- spinless ($\Rightarrow q^\mu$)

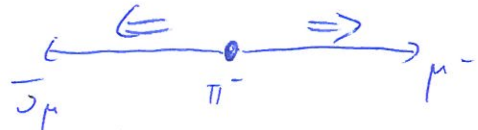
$\longrightarrow$ $\mathcal{M}$ should be Lorentz-invariant ($\Rightarrow$ V or A)

$$\Rightarrow X^\mu = f_\pi(q^2)\, q^\mu = f_\pi\, q^\mu$$

$$\Rightarrow \Gamma(\pi^- \to \mu^- \bar{\nu}_\mu) = \frac{G_F^2}{8\pi} f_\pi^2\, m_\pi\, m_\mu^2 \left(1 - m_\mu^2/m_\pi^2\right)$$

with f_π the „decay constant"

scalar $\pi^- \Rightarrow J=0$



must have pos. hel. $\Rightarrow$ μ^- must have pos. helicity as well!

but: is created as LH : $P_L u_1 \propto m/E$

$\Rightarrow$ • decay impossible if $m=0$

• decay in e^- suppressed, since $m_e \ll m_\mu$

indeed: $$\frac{\Gamma(\pi^-\to e^-\bar{\nu}_e)}{\Gamma(\pi^-\to\mu^-\bar{\nu}_\mu)} = \left(\frac{m_e}{m_\mu}\right)^2\left(\frac{m_\pi^2-m_e^2}{m_\pi^2-m_\mu^2}\right)^2 \sim 10^{-4}$$

decay into heavier particle preferred...

b) Neutrino lepton scattering

$\nu_e e^- \to \nu_e e^-$: guess: $\sigma = G_F^2 s$

fixed target: $s = 2 m_e E_\nu \Rightarrow \sigma = 5\cdot 10^{-17}\,b\left(\frac{E_\nu}{GeV}\right)$

$\left(\text{cf. } \sigma(e^+e^-\to\mu^+\mu^-) \sim \alpha^2/s \sim 10^{-8}\,b\left(\frac{GeV^2}{s}\right)\right)$

correct result: $\frac{d\sigma}{d\Omega}(\nu_e e^-\to\nu_e e^-) = \frac{G_F^2 s}{4\pi^2}$ from $|\overline{\mathcal{M}}|^2 = 16 G_F^2 s^2$

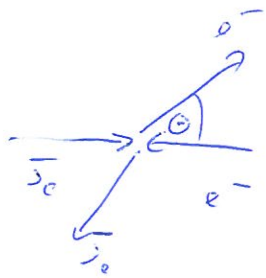
crossing: $\bar{\nu}_e e^- \to \bar{\nu}_e e^-$ has $|\overline{\mathcal{M}}|^2 = 16 G_F^2 t^2 = f(\cos\Theta)$

$$\Rightarrow \frac{\sigma(\nu_e e^-)}{\sigma(\bar{\nu}_e e^-)} = 3$$

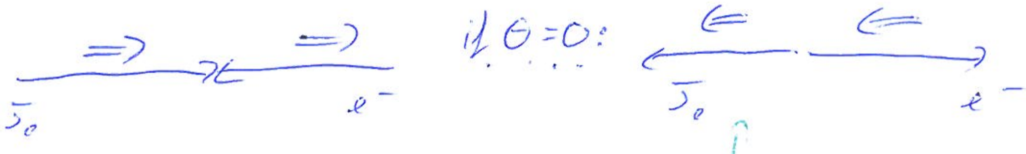
Helicity considerations

$$\frac{d\sigma}{d\Omega}(\bar{\nu}_e e) = 0 \quad \text{for } \Theta = 0$$

(backward scattering)



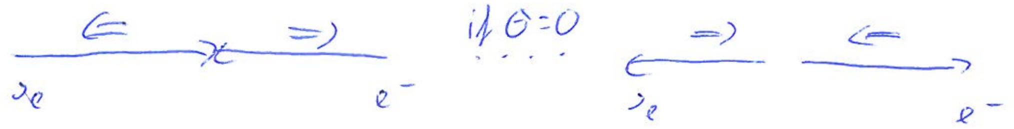
$J_z = 1$
hel. $= +1$



forbidden

the other case:

$J_z = 0$



is fine

moreover: only one of the 3 states is allowed in $\sigma(\bar{\nu}_e e)$
$\Rightarrow$ factor 3 smaller than $\sigma(\nu_e e)$

3) Problems of Fermi-theory

a) Renormalizability

QFT is plagued by infinities, e.g. 

can be solved by redefining the charges in $\mathcal{L}$.

Meaningful only, if a finite number of such redefinitions is enough (there is an infinite number of Feynman graphs that induce infinities)

Such a theory is „renormalizable"

One requirement: dimensionless coupling!

here: $G_F \sim \left(\frac{1}{100\,\text{GeV}}\right)^2$

each diagram with G_F loop is compensated with a Λ^2 infinity to make it have the same dimensions.

$\Rightarrow \Lambda^2, \Lambda^4, \Lambda^6, \ldots$ divergences...

b) Unitarity

$\sigma \simeq G_F^2 s$ grows with s

From unitarity of S-matrix (conserv. of probability):

$$\boxed{\sigma_{tot} = \frac{1}{s} \mathrm{Im}\{ \mathcal{M}(\theta=0)\}} \quad (1) \quad \text{optical theorem}$$

with $\sigma = \int d\Omega \frac{|\mathcal{M}|^2}{64\pi^2 s}$ (2)

partial wave decomposition:

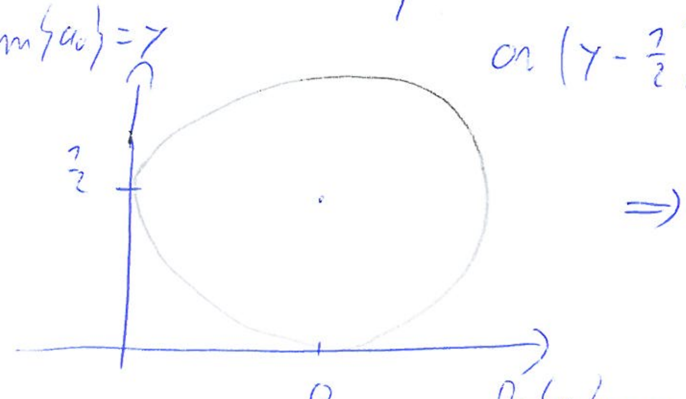
$$\mathcal{M} = 16\pi \sum_{\ell=0}^{\infty} (2\ell+1) \underbrace{P_\ell(\cos\theta)}_{\text{Legendre}} \underbrace{a_\ell}_{\text{partial waves}}$$

insert in (2): $\Rightarrow \sigma = \frac{16\pi}{s} \sum (2\ell+1) |a_\ell|^2$

$$\overset{!}{=} \frac{16\pi}{s} \sum (2\ell+1) \mathrm{Im}\{a_\ell\} \quad \text{(optical theorem)}$$

$\Rightarrow |a_\ell|^2 = \mathrm{Im}\{a_\ell\}$ or $x^2 + y^2 = y$

or $(y - \frac{1}{2})^2 + x^2 = (1/2)^2$



$\Rightarrow |\mathrm{Re}\{a_0\}| < \frac{1}{2}$

For G_F-interaction, point-like, hence $\ell = 0$

$$\Rightarrow \quad \sigma = \frac{16\pi}{s} |a_0|^2 \quad \text{should go with } 1/s$$

but $\sigma = G_F^2 s$ in Fermi-theory

problem becomes actual when

$$\frac{16\pi}{s}\frac{1}{4} = G_F^2 s \Rightarrow s^2 = \frac{4\pi}{G_F^2} \Rightarrow \sqrt{s} \simeq 550\ \text{GeV}$$

Interpretation of Fermi-theory

exchange of massive charged boson W

$$\mathcal{L} = -\frac{g}{2\sqrt{2}}\left[\bar{\nu}_\mu \gamma_\mu (1-\gamma_5)\mu + \bar{\nu}_e \gamma_\mu (1-\gamma_5) e\right] W^{+\mu}$$

$$+ \frac{1}{2} m_W^2 W_\mu^+ (W^{+\mu})^+ + (\partial_\mu W^\mu)(\partial^\mu W_\mu^+)$$

@ low energy $\partial_\mu W^\mu = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial W_\mu^+} = 0$

$$\Rightarrow W^\mu = \frac{g}{\sqrt{2}}\left[\bar{\nu}_\mu \gamma_\mu (1-\gamma_5)\mu + \bar{\nu}_e \gamma_\mu (1-\gamma_5) e\right]\frac{1}{m_W^2}$$

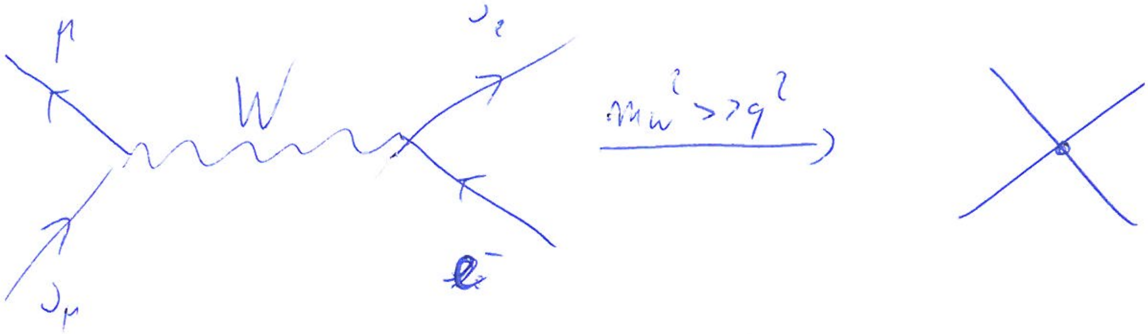
insert back in $\mathcal{L}$

$$\Rightarrow \mathcal{L} = -\frac{g^2}{8 m_W^2}\left[\bar{\nu}_\mu \gamma_\mu (1-\gamma_5)\mu\right]\left[\bar{e}\gamma^\mu (1-\gamma_5)\nu_e\right]$$

$$\Rightarrow \boxed{\frac{g^2}{8 m_W^2} = \frac{G_F}{\sqrt{2}}} \Rightarrow \frac{g}{m_W} \simeq \frac{1}{123\ \text{GeV}}$$

with $m_W \simeq 80$ GeV: $g \simeq 0.65$

(note: $e = \sqrt{4\pi\alpha} \simeq 0.3$)



$$\frac{i\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{m_W^2}\right)}{q^2 - m_W^2} \xrightarrow{m_W^2 >> q^2} \frac{i g^{\mu\nu}}{m_W^2}$$