6.5.13

Recap:

$$\boxed{(i\not\partial - m)\psi(x) = -e\not A\psi(x)}$$

with boundary condition:

$p^0 > 0 : \to$ future (u)

$p^0 < 0 : \to$ past (v)

solved by Green's function

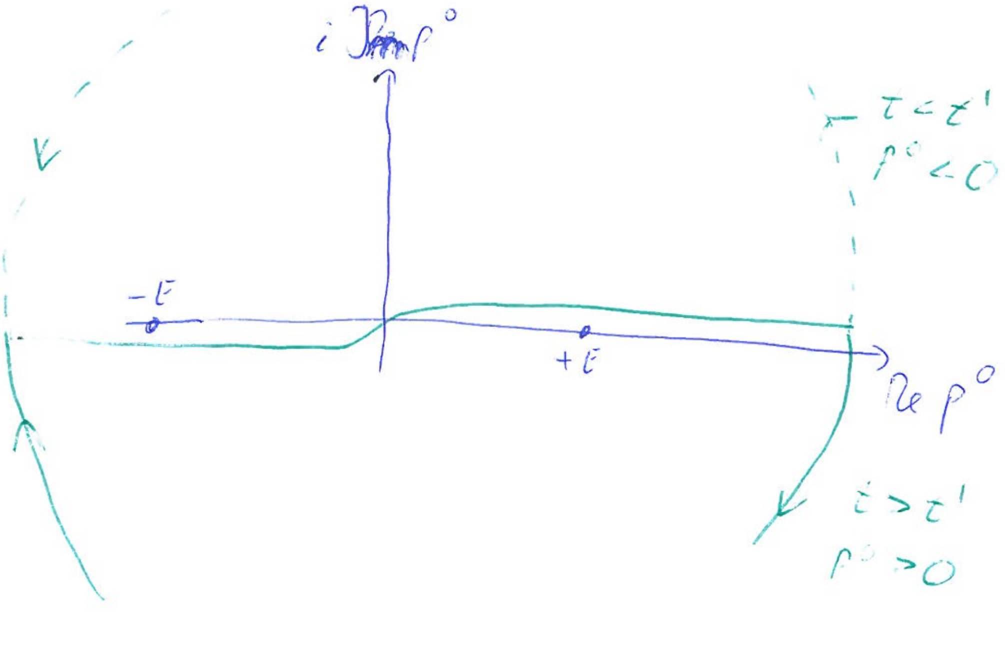
$$\psi(x) = \phi(x) - e\int d^4x'\, K(x-x')\not A(x')\psi(x')$$

ϕ: free solution

$$\psi^{(1)} = \phi(x) - e\int d^4x'\, K(x-x')\not A(x')\phi(x')$$

approximation $e^2 \ll 1$

$$\boxed{\tilde K(p) = \frac{1}{\not p - m}} \; ; \; K(x-x') \propto \int dp^0 \frac{e^{-ip^0(t-t')}(\not p + m)}{(p^0-E)(p^0+E)}$$



e.g.: if $\phi(x') = u(k)e^{-ikx'}$: $\phi(x) = i\int d^3x'\, K(x-x')\gamma^0\phi(x')$ for $t > t'$

$\boxed{\Rightarrow \text{above boundary conditions fulfilled!}}$

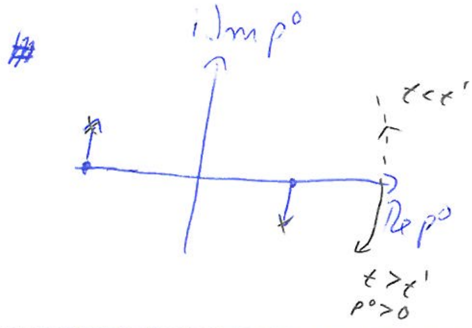
$$\bar\phi(x') = i\int d^3x\, \bar\phi(x)\gamma^0 K(x-x') \quad (t<t') \quad \boxed{0}$$

instead of integration contour, avoid poles via:

$$(p^0 - E)(p^0 + E) \to \left[p^0 + \left(E - \frac{i\varepsilon}{2E}\right)\right]\left[p^0 - \left(E - \frac{i\varepsilon}{2E}\right)\right] = (p^0)^2 - \left(E - \frac{i\varepsilon}{2E}\right)^2$$

$$= p^2 - m^2 + i\varepsilon \qquad (\varepsilon > 0)$$

$\downarrow$ often not explicitely written...



Also: the photon propagator:

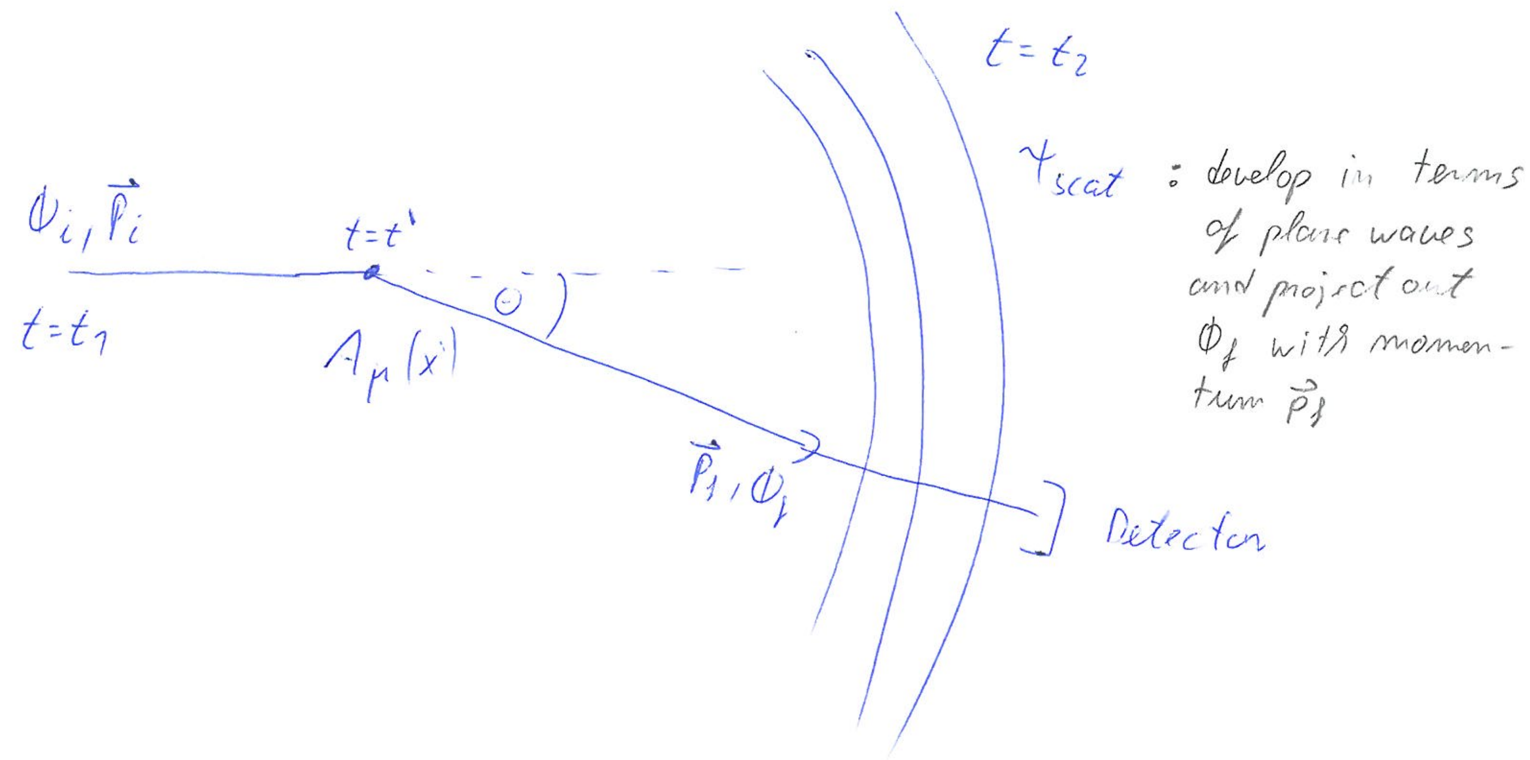
$$\Box A^\mu = e j^\mu = e \bar{\psi}_j \gamma^\mu \psi_i \quad \text{solved by}$$

$$\boxed{A^\mu(x) = e \int d^4x'\, D^{\mu\nu}(x-x')\, j_\nu(x')} \qquad (***)$$

if $\Box D^{\mu\nu}(x-x') = g^{\mu\nu}\delta(x-x')$

$$\xrightarrow{FT} \boxed{D^{\mu\nu}(p) = -\frac{g^{\mu\nu}}{p^2 + i\varepsilon}}$$

III 3) Feynman rules



fermions interact with potential at time t'.

Define S: $\boxed{\Psi_{scat} = S\,\Phi_i}$ with $\Psi^{(1)}_{scat} = \Phi_i(x_2) - e\int d^4x'\, K(x_2 - x') \times A(x')\,\Phi_i(x')$

need to project out Φ_f from Ψ_{scat}: $\int d^3x_2\, \Phi_f^+(x_2)\,\Psi_{scat}$

$$\Rightarrow S^{(1)}_{fi} = \int d^3x_2\, \Phi_f^+(x_2)\, S\, \Phi_i(x_2)$$

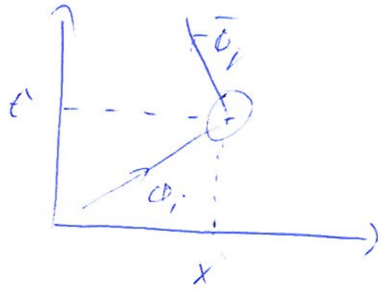
$$= -e\int d^3x_2\, d^4x'\, \underbrace{\Phi_f^+(x_2)\, K(x_2 - x')}_{= -i\,\overline{\Phi}_f(x')} A(x')\,\Phi_i(x')$$

(from (**))

$$\Rightarrow \boxed{S^{(1)}_{fi} = ie\int d^4x'\, \overline{\Phi}_f(x')\, A(x')\, \Phi_i(x')}$$

(selects appropriate solutions with $p^0 \gtrless 0$)

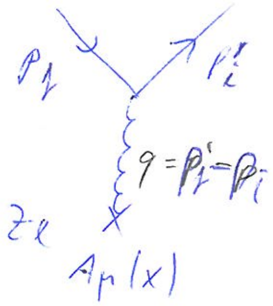
this corresponds to:



interaction with $A_\mu(x')$, integrate over $\int d^4x'$, i.e. over all points where $A_\mu \neq 0$

Example:

1)



$$S^{(1)}_{fi} = ie\int d^4x\, \bar\Phi_f(x)\, \not{A}\, \Phi_i(x)$$

$$\text{"}=\text{"}\; ie\int d^4x\, \bar u(p_f)\, e^{ip_f x}\, \not{A}(x)\, u(p_i)\, e^{-ip_i x}$$

ignore the normalization

$$= ie\,\bar u(p_f')\,\not{A}(q)\,u(p) \equiv -ie\,\bar u\gamma^\mu u\,\frac{-ig_{\mu\nu}}{q^2}\left(-i\,j^\nu(q)\right),$$

with $A_\mu(x) = \left(\frac{Ze}{4\pi|\vec{x}|}, \vec{0}\right) \xrightarrow{FT} \propto \frac{1}{|\vec{q}^{\,2}|} \overset{\dots}{\Longrightarrow} \left(\frac{d\sigma}{d\Omega}\right)_0 = \dots$

no B-field

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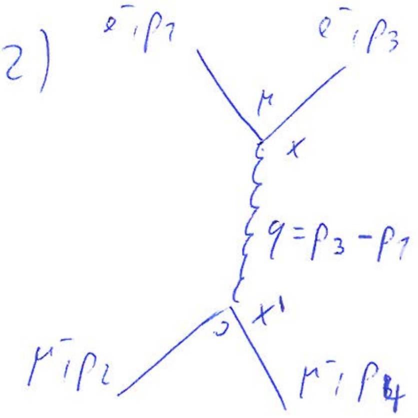
2)



$$A_\mu(x) = e\int d^4x'\,\frac{d^4q}{(2\pi)^4}\,\frac{-g_{\mu\nu}}{q^2+i\varepsilon}\,j^\nu(x')\,e^{-iq(x-x')}$$

$$\text{with } j^\nu(x') = \underbrace{\sqrt{\frac{1}{(2\pi)^3 2E_2}}\,\sqrt{\frac{1}{(2\pi)^3 2E_4}}} \times$$

normalization changed: $V \to (2\pi)^3$

$$\bar u(p_4)\,\gamma^\nu\,u(p_2)\,e^{-i(p_2-p_4)x'}$$

abbreviate $u(p_2) \equiv u_2$ etc.

S_{fi} can be written now as:

$$S_{fi} = \int d^4x\, d^4x' \frac{d^4q}{(2\pi)^4} \left[\frac{1}{(2\pi)^3 2E_1} \frac{1}{(2\pi)^3 2E_2} \frac{1}{(2\pi)^3 2E_3} \frac{1}{(2\pi)^3 2E_4} \right]^{1/2} ie^2$$

$$\bar{u}_3 \gamma^\mu u_1 \frac{-g_{\mu\nu}}{q^2+i\varepsilon} \bar{u}_4 \gamma^\nu u_2\, e^{-iq(x-x')}\, e^{i(p_4-p_2)x'}\, e^{i(p_3-p_1)x}$$

1.) $\int d^4x$: $(2\pi)^4 \delta(p_3 - p_1 - q)$ $\quad q = p_3 - p_1$

2.) $\int d^4x'$: $(2\pi)^4 \delta(p_4 - p_2 + q)$ $\quad q = p_2 - p_4$

} Energy-momentum conservation at each vertex with external lines

3.) $\int d^4q$: $\delta(p_3 - p_1 + p_4 - p_2)$ $\quad p_3 - p_1 = p_2 - p_4$

<u>Note:</u> if momentum not fixed: $\int \frac{d^4k}{(2\pi)^4}$

$$\Rightarrow S_{fi} = -ie^2 \sqrt{}\sqrt{}\sqrt{}\sqrt{}\, (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) \times$$

$$\bar{u}_3 \gamma^\mu u_1 \frac{g_{\mu\nu}}{q^2+i\varepsilon} \bar{u}_4 \gamma^\nu u_2$$

now recall: $S_{fi} = -i(2\pi)^4 \delta(P_f - P_i)\, T_{fi}$

and $T_{fi} = \prod \sqrt{}\sqrt{}\, \mathcal{M}_{fi}$

$\Rightarrow$ heuristic determination of Feynman-rules for $\mathcal{M}$!

here: $\mathcal{M} = -e^2 \bar{u}_3 \gamma^\mu u_1 \frac{g_{\mu\nu}}{q^2+i\varepsilon} \bar{u}_4 \gamma^\nu u_2$

$\Rightarrow \cancel{i\mathcal{M} = -e^2 \, (\bar{u}_3 \gamma^\mu u_1) \, \frac{g_{\mu\nu}}{q^2+i\varepsilon} \, (\bar{u}_4 \gamma^\nu u_2)}$

- overall factor i

- 

u $\qquad\qquad \bar{u}$ $\qquad\qquad v$ $\qquad\qquad \bar{v}$

- 

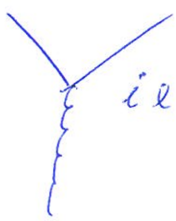
$$\frac{-i g^{\mu\nu}}{q^2 + i\varepsilon}$$

- 

$$\frac{i}{\not{p} - m}$$

- 

ε_μ^* $\qquad\qquad \varepsilon_\mu$

- 

$ie\gamma^\mu$

"extra rules":

- $\int \frac{d^4 q}{(2\pi)^4}$ for closed loops
- (-1) for closed fermion loops
- (-1) relative factor for graphs that differ by interchange of 2 identical fermion lines
- $\frac{1}{n!}$ for n identical particles in final state (Γ or σ multiplied by $\frac{1}{n!}$)

n identical particles, $n!$ ways to count them

but only one is measured

need to evaluate $|\mathcal{M}|^2$ for σ, Γ:

most of the times: $\overline{|\mathcal{M}|}^2$ (average over initial spins a polarizations, sum over final states

$$\Rightarrow q^4 \overline{|\mathcal{M}|}^2 = \frac{e^4}{\underbrace{2\cdot 2}_{\frac{1}{2s_1+1}\cdot\frac{1}{2s_2+1}}} \sum_{s_f, s_i} \left[\bar{u}_3 \gamma^\mu u_1 \, \bar{u}_4 \gamma_\mu u_2\right]\left[u_2^+ \gamma_\nu^+ \bar{u}_4^+ \, u_1^+ \gamma^{\nu +} \bar{u}_3^+\right]$$

use: $\bar{u}^+ = (u^+\gamma_0)^+ = \gamma_0^+ u = \gamma_0 u$

$$= \frac{e^4}{4} \sum \bar{u}_3 \gamma^\mu u_1 \bar{u}_4 \gamma_\mu u_2 \underbrace{u_2^+ \gamma_0}_{\bar{u}_2} \underbrace{\gamma_0 \gamma_\nu^+ \gamma_0}_{\gamma_\nu} u_4 \underbrace{u_1^+ \gamma_0}_{\bar{u}_1} \underbrace{\gamma_0 \gamma^{\nu +} \gamma_0}_{\gamma^\nu} u_3$$

$$= \frac{e^4}{4} \sum \bar{u}_3 \gamma^\mu u_1 \bar{u}_4 \gamma_\mu \underbrace{u_2 \bar{u}_2}_{\not p_2 + m_2} \gamma_\nu u_4 \bar{u}_1 \gamma^\nu u_3$$

$$= \frac{e^4}{4} \sum \underline{(\bar{u}_3)_\alpha} (\gamma^\mu)_{\alpha\beta} (u_1)_\beta (\bar{u}_4)_\gamma (\gamma_\mu)_{\gamma\delta} (\not p_2 + m_2)_{\delta\varepsilon} (\gamma_\nu)_\varepsilon (u_4)_\eta (\bar{u}_1)_\sigma (\gamma^\nu)_{\sigma\tau} \underline{(u_3)}$$

$$= \frac{e^4}{4} \operatorname{Tr}\left\{\gamma^\mu (\not p_1 + m_1) \gamma^\nu (\not p_3 + m_3)\right\} \operatorname{Tr}\left\{\gamma_\mu (\not p_2 + m_2) \gamma_\nu (\not p_4 + m_4)\right\}$$

$\Rightarrow$ Trace theorems

$$\mathrm{Tr}\,\mathbb{1} = 4$$

$$\mathrm{Tr}\{\gamma_\mu\} = \mathrm{Tr}\{\gamma_5\} = 0 \qquad \text{where } \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$$

$$\gamma_5^2 = \mathbb{1}\ ;\ \gamma_5^\dagger = \gamma_5$$

$$\{\gamma_5, \gamma_\mu\} = 0$$

$$\mathrm{Tr}\{\gamma^\mu\gamma^\nu\} = 4g^{\mu\nu} \Rightarrow \mathrm{Tr}\{\not a \not b\} = 4\, a\cdot b$$

$$\mathrm{Tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} = 4\left(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}\right)$$

$$\vdots$$

$$\Rightarrow q^4\,|\overline{M}|^2 = 4e^4\left[2(p_1\cdot p_2)(p_3\cdot p_4) + 2(p_1\cdot p_4)(p_2\cdot p_3) + q^2\left(q^2 + (p_1\cdot p_3) + (p_2\cdot p_4)\right)\right]$$

$$\Rightarrow |\overline{M}|^2 \overset{m_1^2=0}{=} 2e^4(s^2+u^2)/t^2$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega}(e^-\mu^- \to e^-\mu^-) = \frac{\alpha^2}{2}\,\frac{s^2+u^2}{s\,t^2}} \qquad \alpha = \frac{e^2}{4\pi}$$

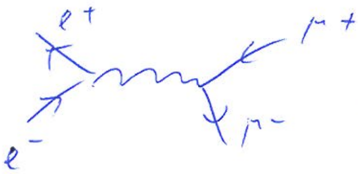
note: $q^2 = t = -2p_1\cdot p_3 = -2p_2\cdot p_4$

$s = 2p_1\cdot p_2 = 2p_3\cdot p_4$

$u = -2p_1\cdot p_4 = -2p_2\cdot p_3$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\,|\overline{M}|^2\sqrt{\frac{\lambda(s, m_1^2, m_2^2)}{\lambda(s, m_3^2, m_4^2)}}$$

Crossing: $e^+ e^- \longrightarrow \mu^+ \mu^-$



$(s \leftrightarrow t)$ in $|\mathcal{M}|^2$!

$$\Rightarrow \quad \frac{d\sigma}{d\Omega}(e^+e^- \to \mu^+\mu^-) = \frac{\alpha^2}{2}\,\frac{t^2+u^2}{s^3}$$

it follows: $\frac{d\sigma}{d\Omega}(e^+e^- \to \mu^+\mu^-) = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right)$

$$\begin{pmatrix} t = -\frac{s}{2}(1-\cos\theta) \\ u \qquad (\;+\;\;) \end{pmatrix}$$

$$\Rightarrow \quad \sigma = \frac{4\pi\alpha^2}{3s} \simeq \frac{87\,\mathrm{nb}}{s/\mathrm{GeV}^2}$$