29.4.13

III QED, QFT, Feynman rules

III 1) How to describe fermions and photons

Dirac-eq. for free particles:

$$\boxed{(i\not{\partial} - m)\psi = (\not{p} - m)\psi = 0}$$

+ adjoint spinor $\bar{\psi} = \psi^{\dagger}\gamma_0$: $(\not{p} + m)\bar{\psi} = 0$

↳ conserved current: $\partial_\mu j^\mu = 0$ with $j^\mu = \bar{\psi}\gamma^\mu\psi$

4 solutions (2 E>0, 2 E<0), can be written (E>0 for all) as:

$$\psi_{1,2} = e^{-ipx}\, u_{1,2}(\vec{p}) \qquad (\not{p} - m)u = \bar{u}(\not{p} - m) = 0$$

$$\psi_{3,4} = e^{ipx}\, v_{2,1}(\vec{p}) \qquad (\not{p} + m)v = \bar{v}(\not{p} + m) = 0$$

$[(-\not{p} - m)\, u(-\vec{p}) = 0]$

Normalize to

$$\boxed{\begin{aligned} &\bar{u}_s u_{s'} = 2m\,\delta_{ss'} = -\bar{v}_s v_{s'} \\ &\sum_s u_s \bar{u}_s = \not{p} + m \\ &\sum_s v_s \bar{v}_s = \not{p} - m \\ &u_s^{\dagger} u_{s'} = v_s^{\dagger} v_{s'} = 2E\,\delta_{ss'} \end{aligned}}$$

check with

$$u_s = \sqrt{E+m}\begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\vec{p}}{E+m}\chi_s \end{pmatrix}$$

$$v_s = \sqrt{}\begin{pmatrix} \frac{\vec{\sigma}\vec{p}}{E+m}\phi_s \\ \phi_s \end{pmatrix}$$

$\chi_1 = \chi_{1/2} = \begin{pmatrix}1\\0\end{pmatrix} = \phi_2$

$\chi_2 = \chi_{-1/2} = \begin{pmatrix}0\\1\end{pmatrix} = \phi_1$

where u_1 has spin $\parallel \vec{p}$; same as v_1 ; (up)

u_2 " anti-$\parallel \vec{p}$; same as v_2 ; (down) most general solution:

$$\overset{(-)}{\psi}(x) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3 2E}} \left[a_s^{(*)}(p)\, \overset{(-)}{u_s}\, e^{(\mp) ipx} + b_s^{*()}\, \overset{(-)}{v_s}\, e^{(\pm) ipx} \right]$$

$V = 1$ with $\int \psi^+ \psi d^3x = 1$

QFT (2nd quantization): $a \to \hat{a}$ $\quad \psi \to \hat{\psi}$ Operators

$b^+ \to \hat{b}^+$

$\hat{a}$: annihilates particle

$\hat{b}^+$: creates antiparticle

Interaction with photon field:

„minimal substitution" $\quad p^\mu \to p^\mu - qA^\mu$

„covariant derivative" $\quad \partial^\mu \to \partial^\mu + iqA^\mu$

$\left(p^\mu = i\partial^\mu, \; q(e^-) = -e; \; e > 0 \right)$

$\Rightarrow (i\not{\partial} - q\not{A} - m)\psi = 0$ or, in Lagrangian:

$$\boxed{\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi - q\bar{\psi}\gamma^\mu A_\mu \psi}$$

field strength tensor $\quad$ free particle $\quad$ interaction

Euler Lagrange: $\partial_\mu F^{\mu\nu} = q j^\nu \longrightarrow \partial_\nu j^\nu = 0$

QED: theory of $\mathcal{L}$

$$A_\mu = \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} \sum_{\lambda=0}^{3} \varepsilon_\mu(k,\lambda)\left[a(k,\lambda)e^{-ikx} + a^\dagger(k,\lambda)e^{ikx}\right]$$

with $\sum_{\lambda=1}^{2} \varepsilon_\mu^* \varepsilon_\nu = -g_{\mu\nu}$ for real photons (Exercise)

III 2) How to propagate fermions and photons

$$\boxed{(i\not{\partial} - m)\psi(x) = -e\not{A}\psi(x)} \qquad (*)$$

how to solve?

Reminder electrodynamics: Poisson: $\vec{\nabla}^2 \phi = -\rho(\vec{x})$

potential   charge density

for point charge $\rho(\vec{x}) = q\,\delta(\vec{x}-\vec{x}')$: $\phi(\vec{x}) = \frac{q}{4\pi|\vec{x}-\vec{x}'|}$

for continuous charge distribution: integrate over potentials of individual charges: $\phi(x) = \int d^3x' \frac{\rho(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|}$

alternative derivation: introduce Green's function

definition: $\vec{\nabla}^2 G(\vec{x},\vec{x}') = -\delta(\vec{x}-\vec{x}')$

inh. diff. eq. with boundary conditions

related to transform function's boundary value to the function's value at some point.

Setting $\Phi(x) = \int d^3x' \, G(\vec{x}, \vec{x}') \, \rho(\vec{x}')$, solves Poisson:

$$\vec{\nabla}^2 \Phi = -\rho(\vec{x}) \Rightarrow G(\vec{x}, \vec{x}') = G(\vec{x} - \vec{x}') = \frac{1}{4\pi |\vec{x} - \vec{x}'|}$$

$$-\mathcal{L} = \rho(\vec{x}) \Phi(\vec{x}) = \int d^3x \, \rho(\vec{x}) \, G(x - \vec{x}') \rho(\vec{x}') \qquad \underset{\rho(x)}{\circ} \overset{G}{\text{—}} \underset{\rho(x')}{\circ}$$

Same approach now for $(*)$:

Define $(i\not{\partial} - m) K(x, x') = \delta(x - x')$ as Green's function

For known $K(x, x')$ the solution of $(*)$ is:

$$\boxed{\psi(x) = -e \int d^4x' \, K(x, x') \not{A}(x') \psi(x') + \underbrace{\Phi(x)}_{\text{free solution}}}$$

proof:

$$(i\not{\partial} - m)\psi(x) = (i\not{\partial} - m)\left[\Phi(x) - e\int d^4x' \, K(x - x') \not{A}(x') \psi(x')\right]$$

$$= -e \int d^4x' \, (i\not{\partial} - m) K(x - x') \not{A}(x') \psi(x')$$

$$= -e \not{A}(x) \psi(x)$$

Choose approximative approach: $(e^2 = 4\pi\alpha \ll 1)$

$$\psi^{(1)}(x) = \Phi(x) - e \int d^4x' \, K(x - x') \not{A}(x') \Phi(x')$$

$$\psi^{(2)}(x) = \Phi(x) - e\int d^4x' \, K(x - x') \not{A}(x') \Phi(x') + e^2 \int d^4x'' d^4x' \, K(x - x'') \times$$

$$\times \not{A}(x'') K(x'' - x') \not{A}(x') \Phi(x')$$

$\psi^{(3)}(x) = \dots$

useful: Fourier-trafo of K

$$K(x-x') = \frac{1}{(2\pi)^4} \int d^4p \; \tilde{K}(p) \, e^{-ip(x-x')}$$

insert in definition of $K(x-x')$:

$$(i\not{\partial} - m) K(x-x') = \frac{1}{(2\pi)^4} \int d^4p \, (\not{p} - m) \tilde{K}(p) \, e^{-ip(x-x')}$$

$$\stackrel{!}{=} \delta(x-x')$$

$$\Rightarrow \boxed{\tilde{K}(p) = \frac{1}{\not{p} - m} = \frac{\not{p} + m}{p^2 - m^2}}$$ Electron propagator

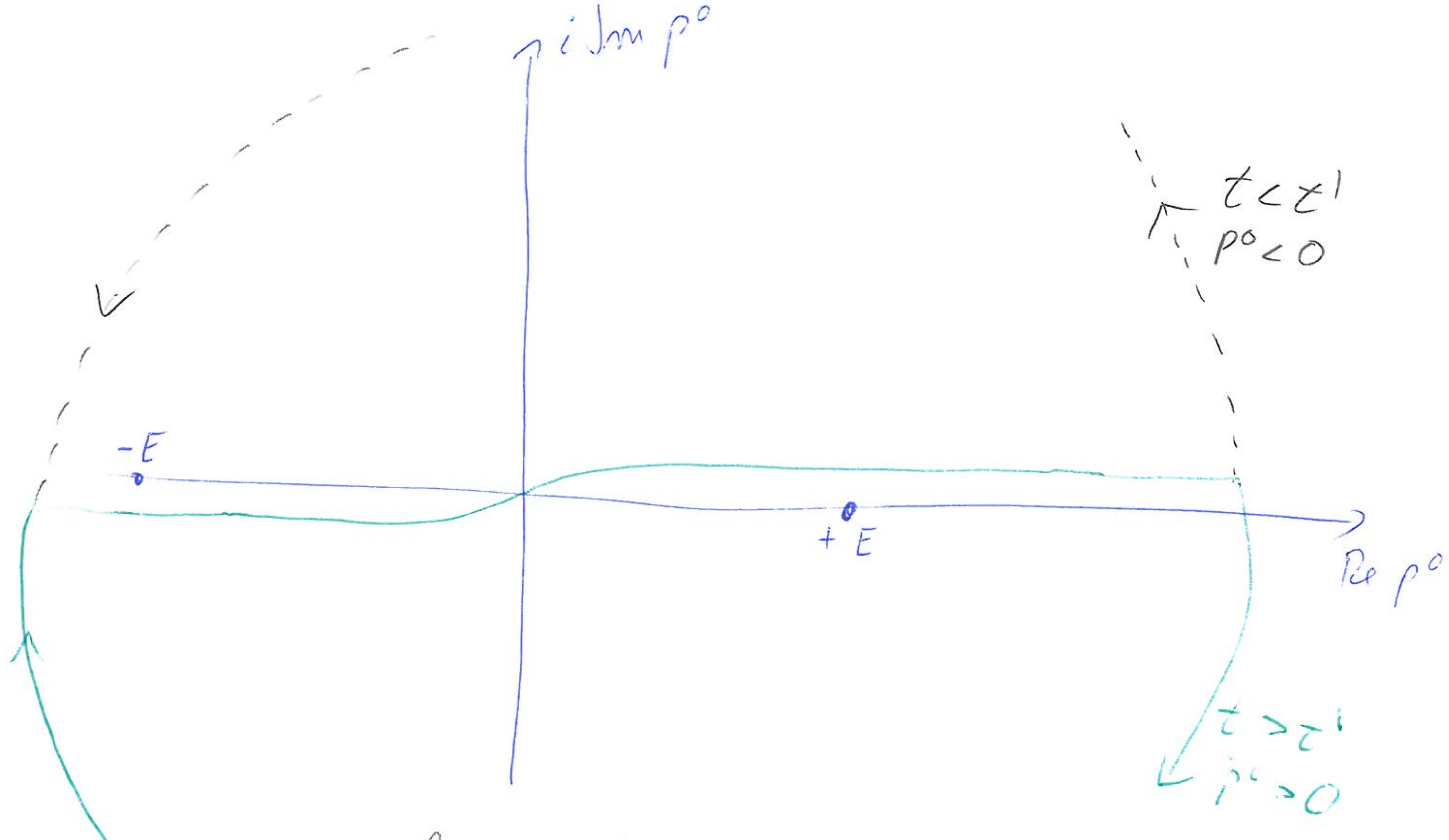
rewrite:

$$K(x-x') = \frac{1}{(2\pi)^4} \int d^3p \, e^{i\vec{p}(\vec{x}-\vec{x}')} \int_{-\infty}^{\infty} dp^0 \frac{e^{-ip_0(t-t')}(\not{p}+m)}{(p^0 - E)(p^0 + E)}$$

with p^0 independent of $E = +\sqrt{\vec{p}^{\,2} + m^2}$
($\leftrightarrow$ virtual electrons, $p^2 \neq m^2$)

due to poles at $p^0 = \pm E$, the integral does not converge...
$\Rightarrow$ need to specify the <u>boundary conditions</u>:

want: $\left\{ \begin{array}{l} \text{solutions with } t > t' \text{ have } p^0 > 0 \\ \text{solutions with } t < t' \text{ have } p^0 < 0 \end{array} \right\} \leftrightarrow$ Feynman-Stückelberg to achieve this, we change the integration contour to C:



complex analysis „Funktionentheorie" → radius $\to \infty$ Lemma of Jordan

$\Rightarrow e^{-ip^0(t-t')} \to 0$ with $\mathrm{Im}\, p^0$ large

[recall residue theorem: $\oint_C f(z)\, dz = 2\pi i \lim_{a \to a_k} (a - a_k) f(a_k)$

closed path first order poles]

$$\Rightarrow \text{for } t > t': \int dp^0 \frac{1}{p^0 - E} \frac{(\not{p} + m) e^{-ip^0(t-t')}}{p^0 + E}$$

$$= -2\pi i \left.\frac{(\not{p} + m) e^{-ip^0(t-t')}}{p^0 + E}\right|_{p^0 = +E}$$

$$\Rightarrow \boxed{K(x-x') = -i(2\pi)^{-3}\int d^3p\, e^{-i[E(t-t') - \vec p(\vec x - \vec x')]}\,\frac{\gamma^0 E - \vec\gamma\vec p + m}{2E} \quad \text{for } t>t'}$$

$$\boxed{K(x-x') = -i(2\pi)^{-3}\int d^3p\, e^{i[E(t-t') + \vec p(\vec x - \vec x')]}\,\frac{-\gamma^0 E - \vec\gamma\vec p + m}{2E} \quad \text{for } t<t'}$$

this in fact satisfies our boundary conditions!

consider free particle with k^μ, $k^0>0$: $\psi(x') = u(k)\,e^{-ikx'}$

at time $t>t'$: $\boxed{\psi(x) = i\int d^3x'\, K(x-x')\,\gamma^0\,\psi(x')}$

proof: $\text{rhs} = \underline{(2\pi)^{-3}}\int d^3p\, \underline{d^3x'}\, e^{-i[E(t-t') - \underline{\vec p}(\vec x - \underline{\vec x'})]}\, e^{-ik^0t' + \underline{i\vec k\vec x}}$

$$\frac{\gamma^0 E - \vec\gamma\vec p + m}{2E}\,\gamma^0 u(k)$$

$$= \int d^3p\, \underline{\delta(\vec k - \vec p)}\, \exp\left\{-i\left[E(t-t') - \vec p\vec x + k^0 t'\right]\right\} \times$$

$$\frac{\gamma^0 E - \vec\gamma\vec p + m}{2E}\,\gamma^0 u(k) \quad \left(\overset{\text{the } \delta \text{ leads to:}}{\vec k = \vec p \Rightarrow k^0 = E^0}\right)$$

$$= \exp\left\{-i[k^0 t - \vec k\vec x]\right\}\,\frac{\gamma^0 k^0 - \vec\gamma\vec k + m}{2k^0}\,\gamma^0 u(k)$$

$$= e^{-ikx}\,u(k)$$

with: $(\gamma^0 k^0 - \vec\gamma\vec k + m)\gamma^0 u = \gamma^0(\gamma^0 k^0 + \vec\gamma\vec k + m)u$ and $(\gamma^0 k^0 - \vec\gamma\vec k - m)u = 0$

it follows $\gamma^0(\gamma^0 k^0 + \vec\gamma\vec k + m)u = 2k^0 u$

in analogy: for $t < t'$, there is a term

$$(-\gamma^0 E - \vec{\gamma}\vec{p} + m)\gamma^0 u(p) = 0$$

$$\Rightarrow \boxed{i\int d^3x' \, K(x-x')\gamma^0 \Phi(x') = \begin{cases} \Phi(x) & t > t' \\ 0 & t < t' \end{cases}}$$

and (try @ home!)

$$\boxed{i\int d^3x \, \bar{\Phi}\gamma^0 K(x-x') = \begin{cases} \bar{\Phi}(x') & t' < t \\ 0 & t' > t \end{cases}} \quad (**)$$

plane wave with positive k^0 only propagated in future

$\overline{\text{plane wave}}$ with ~~negative~~ positive k^0 only propagated in past

plane wave with negative k^0 only propagated in past

$\overline{\text{plane wave}}$ with negative k^0 only propagated in future