Recap:

Feynman-Stückelberg:

positron emitted $\left.\begin{matrix}+\vec{p}\\ +E\end{matrix}\right\} +p^\mu, p^0>0$ electron absorbed $\left.\begin{matrix}-\vec{p}\\ -E\end{matrix}\right\} -p^\mu, p^0<0$

make sense of wave functions with neg. energy by letting them go backwards in time.

Also: absorption of antiparticle with $\left.\begin{matrix}+\vec{p}\\ +E\end{matrix}\right\} +p^\mu, p^0>0$

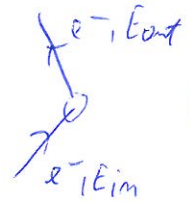
is equivalent to emission of particle with

$\left.\begin{matrix}-\vec{p}\\ -E\end{matrix}\right\} -p^\mu, p^0<0$

Example: e^+ a e^- and time-dependent $V = V_0\, e^{-i\omega t}$ (gives away energy).

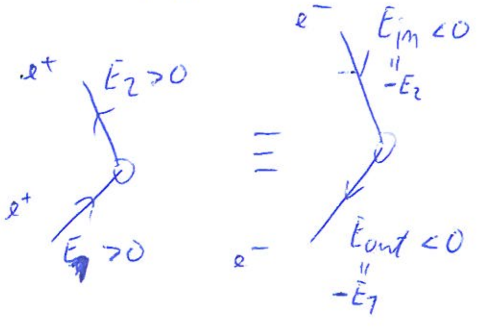
call e^- „the particle":

(a)



$\Omega = \int \psi^*_{out} V \psi_{in}\, dt \rightarrow \delta(E_{out} - E_{in} - \omega) \Rightarrow E_{out} = E_{in} + \omega$

(b)

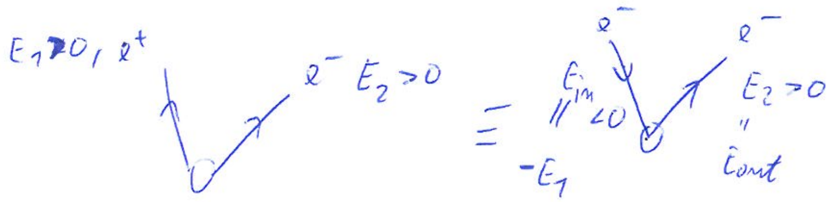


$\Omega = \int \psi^*_{out} V \psi_{in} \rightarrow \delta(E_{out} - E_{in} - \omega)$

$\rightarrow \delta(-E_1 + E_2 - \omega) \Rightarrow E_2 = E_1 + \omega$

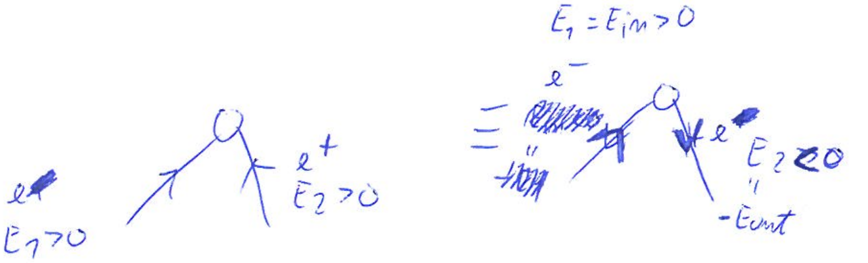
$\Rightarrow$ outgoing e^+ has gained energy ✓

(c) creation of e^+e^--pair



$\Omega \propto \dots \longrightarrow \delta(E_{out} - E_{in} - \omega) \longrightarrow \delta(E_2 + E_1 - \omega) \Rightarrow E_1 + E_2 = \omega$

(d) annihilation of e^+e^- pair (need to choose $V = V_0\, e^{+i\omega t}$)



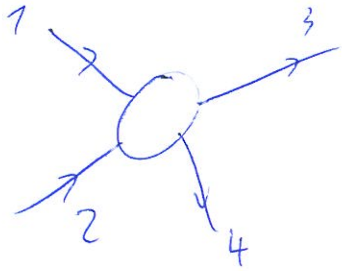
$\delta(E_{out} - E_{in} + \omega)$

$\hookrightarrow \delta(-E_2 - E_1 + \omega)$

$\Rightarrow \omega = E_1 + E_2$ ✓

$\Rightarrow$ same formalismus ~~with~~ for different processes (e^-, e^+, e^-e^+, e^-e^+)

Crossing symmetry



$1 + 2 \to 3 + 4$

„blob" described by analytical function $T_s(s, t, u)$.

e.g. $e^+ e^- \to \mu^+ \mu^-$

Turns out: same analytical function describes

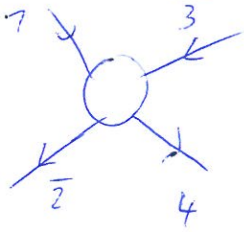
$1 + \bar{3} \to \bar{2} + 4$ $\quad$ $e^+ \mu^- \to e^+ \mu^-$

$1 + \bar{4} \to \bar{2} + 3$ $\quad$ $e^+ \mu^+ \to e^+ \mu^+$

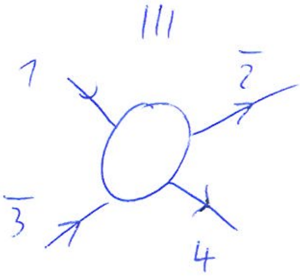
with negative energies for antiparticles.

↳ $\boxed{\text{$T(s,t,u)$ can be analytically continued to negative energies. (or: from } \begin{smallmatrix} s \geq 0 \\ t,u \leq 0 \end{smallmatrix} \text{ to } \begin{smallmatrix} t \geq 0 \\ u,s \leq 0 \end{smallmatrix} \text{ etc)}}$

E.g.



$p_1 + p_{\bar{3}} = p_{\bar{2}} + p_4$



always defined like this

p_3^0 now $-p_3^0$

$t \equiv (p_1 - p_3)^2 = (p_1 + p_{\bar{3}})^2 > 0$

$s \equiv (p_1 + p_2)^2 = (p_1 - p_{\bar{2}})^2 < 0$

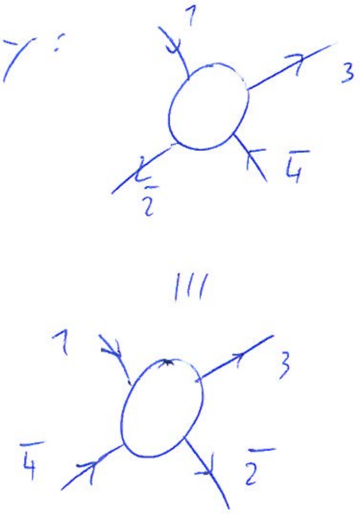
u as before

$\Rightarrow T_t(s, t, u)$ mit $t > 0,\ s, u \leq 0$

$= T(s, t, u)$

$\boxed{\text{t-channel diagram}}$

In analogy:



$p_1 + p_{\bar 4} = p_{\bar 2} + p_3$

$$u \equiv (p_1 - p_4)^2 = (p_1 + p_{\bar 4})^2 > 0$$

$$s \equiv (p_1 + p_2)^2 = (p_1 - p_{\bar 2})^2 \leq 0$$

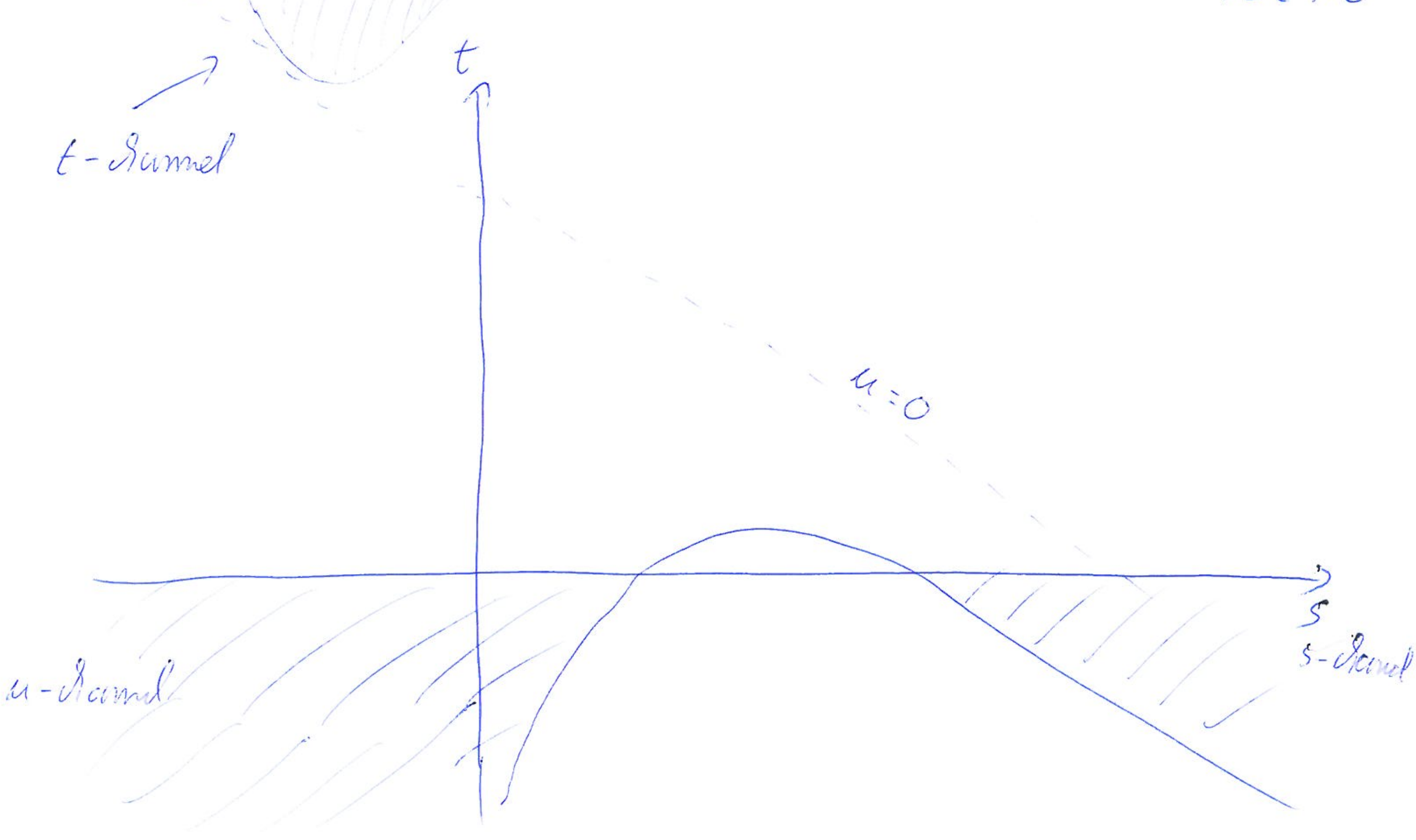
t as before

$\boxed{\text{u-channel diagram}}$

$$T_u(s,t,u) = T(s,t,u) \quad \text{with } u > 0, \; s,t \leq 0$$

In practice: Call the positive variable s

$$\Rightarrow T(e^+\mu^- \to e^+\mu^-) = T(e^+e^- \to \mu^+\mu^-)\Big|_{s \leftrightarrow t}$$



Example: $e^- e^- \to e^- e^-$ s-channel

$\Rightarrow$ $e^- e^+ \to e^- e^+$ is t-channel

obtained from s-channel by $s \leftrightarrow t$

u-channel: $e^- e^+ \to e^- e^+$ $\Rightarrow$ obtained from s-channel by $s \leftrightarrow u$

but: is the same process as t-channel!

$\Rightarrow$ expect a $t \leftrightarrow u$ symmetry

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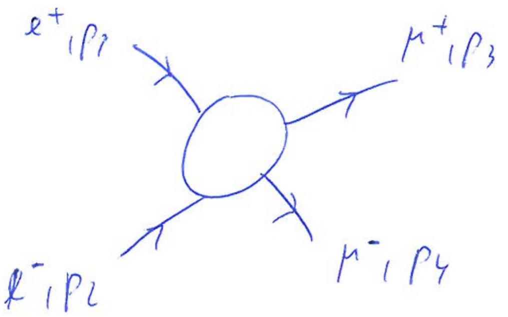
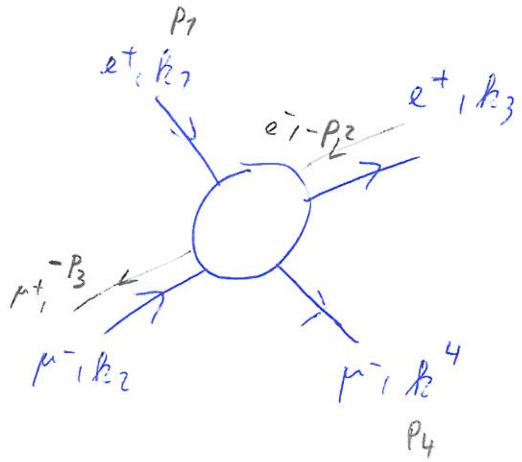
maybe more intuitive:





$p_1 \longrightarrow k_1$

$p_2 \longrightarrow -k_3$

$p_3 \longrightarrow -k_2$

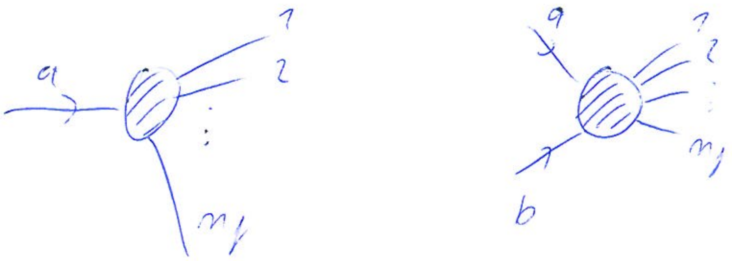
$p_4 \longrightarrow k_4$

$$\Rightarrow \quad s = (p_1+p_2)^2 \longrightarrow (k_1-k_3)^2 = t$$

$$t = (p_1-p_3)^2 \longrightarrow (k_1+k_2)^2 = s$$

$$u = (p_1-p_4)^2 \longrightarrow (k_1-k_4)^2 = u$$

II Cross section and phase space



blob: later (dynamics, theory)

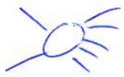
Introduce S-Matrix $\boxed{\langle f | S | i \rangle \equiv S_{fi}}$

connected to H': $i \frac{\partial}{\partial t} |t\rangle = \hat{H}'(t) |t\rangle$

$\lim_{t \to -\infty} |t\rangle = |i\rangle$"
$\quad\; {}_{+\infty} \qquad\quad |f\rangle$

(unitary)

$$S_{fi} = \delta_{fi} - i (2\pi)^4 \, \delta(P_f - P_i) \, T_{fi}$$

= (ignore this from now on) + 

Transition matrix

II 1) Transition rate, cross section, decay rates

$$\boxed{d W_{fi} = \frac{|S_{fi}|^2}{T} \, d N_f}$$

sum over all final states, integrate over all momenta

$$= \prod_{f=1}^{n_f} \frac{d^3 p_f}{(2\pi)^3} \frac{V}{2 E_f}$$

No. of states (standing waves) in interval $\vec{p} \to \vec{p} + d\vec{p}$ in a box of volume $V \leftrightarrow dn = \frac{L}{2\pi} dp$

from $\psi(x+L) = \psi(x)$

per. boundary cond. $\leftrightarrow$ no flow out of V

particle wave function norm: $2E$ in V

$\hookrightarrow \psi \propto N e^{-ipx}$: $N = \sqrt{\frac{1}{2EV}}$

there are $2E$ particles in V

note that ~~[illegible]~~ $|S_{fi}|^2 = (2\pi)^8 \, |T_{fi}|^2 \underbrace{\left[\delta(P_f - P_i)\right]^2}_{= VT\,\delta(P_f - P_i)}$

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the trick from Fermi's Golden rule is applied:

$$\left[\delta(P_f - P_i)\right]^2 = \delta(P_f - P_i) \int_{-T/2}^{T/2} \int_{-V/2}^{V/2} dt\, d^3x \; e^{-i(E_i - E_f)t} \, e^{-i(\vec{p}_f - \vec{p}_i)\vec{x}}$$

$$= \delta(P_f - P_i) \int_{-T/2}^{T/2} \int_{-V/2}^{V/2} dt\, d^3x = \delta(P_f - P_i)\, V T\,,$$

where we have used: $\delta(x)\, g(x) = \delta(x)\, g(0)$

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with defining $T_{fi} = \prod_i \sqrt{\frac{1}{2E_iV}} \prod_{f=1}^{n_f} \sqrt{\frac{1}{2E_fV}} \; M_{fi}$:

$$\boxed{dw_{fi} = \frac{V^{1-n_i}}{(2\pi)^{3n_f - 4}} \, \delta(P_f - P_i) \, |M_{fi}|^2 \prod_i \frac{1}{2E_i} \prod_f \frac{d^3p_f}{2E_f}}$$

recall / note that $\int \frac{d^3p_f}{2E_f}$ is Lorentz-invariant:

$$\int d^4p \, \delta(p^2 - m^2)\, \Theta(E) = \int d^3p \int dE \, \delta(E^2 - \tilde{\vec{p}}^{\,2})\, \Theta(E) \qquad (\tilde{\vec{p}}^{\,2} = \vec{p}^{\,2} + m^2)$$

$$= \int d^3p \int \frac{dE}{2|\tilde{\vec{p}}|} \left[\delta(E - |\tilde{\vec{p}}|) + \delta(E + |\tilde{\vec{p}}|)\right] \Theta(E)$$

$$= \int \frac{d^3p}{2E}$$

A) Decay

$$\boxed{\Gamma = \frac{1}{2E_a} \frac{1}{(2\pi)^{3n_f-4}} \int \frac{d^3p_1}{2E_1} \cdots \frac{d^3p_{n_f}}{2E_{n_f}} |M_{fi}|^2 \delta(P_f - P_i)}$$

$E_a = M$, life time is $\boxed{\tau \equiv \frac{1}{\Gamma}}$

$\Gamma = -\frac{dN_a}{dt} \frac{1}{N_a}$

$\Rightarrow N_a(t) = N_a(0)\, e^{-\Gamma t}$

$\frac{1}{2E_a}$: # of decaying particles per unit volume

B) Cross section

$$\sigma \equiv \frac{\text{\# transitions per time}}{\text{\# incoming particles per area and time} \equiv \text{flux}}$$

flux = density of incoming particles $\times$ velocity relative to target

$$= \frac{1}{V} |\vec{v}_a - \vec{v}_b| = \frac{1}{V} \frac{1}{E_a E_b} \sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}$$

$$\Rightarrow \boxed{\sigma = \frac{1}{4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}} \frac{1}{(2\pi)^{3n_f-4}} \int \frac{d^3p_1}{2E_1} \cdots \frac{d^3p_{n_f}}{2E_{n_f}} |M_{fi}|^2 \delta(P_f - P_i)}$$

Note: this is CMS. In LS: $\prod_i \frac{1}{2E_i} = \frac{1}{2E_a} \frac{1}{2M}$

$$\text{flux} = \frac{1}{V} |\vec{v}_a| = \frac{|\vec{p}_a|}{E_a} \frac{1}{V}$$

II 2) 2-2 Cross section

$$(2\pi)^2 \, d\Phi_2 = \int \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2} \, \delta(p_a + p_b - p_1 - p_2)$$

$$= \dots = \int d\Omega \, \frac{|\vec{p}_1|}{4\sqrt{s}}$$ Exercise

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \sqrt{\frac{\lambda(s, m_1^2, m_2^2)}{\lambda(s, m_a^2, m_b^2)}} \, |M_{fi}|^2}$$

$$\boxed{\frac{d\sigma}{dt} = \frac{1}{16\pi \, \lambda(s, m_a^2, m_b^2)} \, |M_{fi}|^2}$$

also useful: $$\boxed{\frac{d\Gamma}{d\Omega} = \frac{1}{64\pi^2} \frac{\sqrt{\lambda(m_a^2, m_1^2, m_2^2)}}{m_a^3} \, |M_{fi}|^2}$$

for $a \to 1 + 2$ End 22.4.

typically:

- $|\overline{M}_{fi}|^2$: averaged over initial spins, summarized over final spins / polarizations

- n identical particles in final state
  multiply Γ, σ with $\frac{1}{n!}$

  n identical particles: $n!$ possibilities to count ~~them~~ arrange them, but only one is measured