Time-Traveling Sterile Neutrinos

(Adventures in Extra-Dimensions)

Tom Weiler
Vanderbilt University

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Syllabus:

* Goedel, van Stockum,Tipler- planes
* Spacetime metrics, warped space
* Closed Timelike Curves (CTCs)
* From 5D to 6D
* Energy distribution
* Conclude

GTvS spacetime

Gödel metric describes a pressure-free perfect fluid with negative cosmological constant and rotating matter, and the Tipler-van-Stockum (TvS) spacetime is being generated by a rapidly rotating infinite cylinder. In both cases the metric can be written as

$$ds^2 = +g_{tt}(r)\,dt^2 + 2g_{t\phi}(r)\,dt\,d\phi - g_{\phi\phi}(r)\,d\phi^2 - g_{rr}\,dr^2 - g_{zz}\,dz^2\,. \qquad (1)$$

Distortions of the “lengths” phi and t
in the radial direction
is an example of warping.

And an example of time-warping is
the Robertson-Walker big-bang metric.

Negative time:

A dynamical approach to GTvS causality examines the purely azimuthal null-curve with $ds^2 = 0$. One gets

$$\dot{\phi}_\pm = \frac{g_{t\phi} \pm \sqrt{g_{t\phi}^2 + g_{tt}\, g_{\phi\phi}}}{g_{\phi\phi}}, \tag{3}$$

where the $\pm$ refers to co-rotating and counter-rotating lightlike signals. The coordinate time for a co-rotating path is

$$\Delta T_+ = \Delta\phi \left(\frac{g_{\phi\phi}}{g_{t\phi} + \sqrt{g_{t\phi}^2 + g_{\phi\phi}\, g_{tt}}} \right) . \tag{4}$$

As $g_{\phi\phi}$ goes from positive to negative, the light-cone tips such that the azimuthal closed path is traversed in negative time

$$\Delta T_+ = \frac{-2\pi\, |g_{\phi\phi}|}{g_{t\phi} + \sqrt{g_{t\phi}^2 + g_{\phi\phi}\, g_{tt}}} \qquad [g_{\phi\phi} < 0] \, . \tag{5}$$

The quantum returns to its origin before it left, marking the existence of a CTC.

Note that the Lorentzian signature is maintained even as $g_{\phi\phi}$ switches sign as long as the argument of the square-root, proportional to $-g_4$, remains positive definite. Note also that the Lorentzian signature is maintained as $g_{\phi\phi}$ switches to a negative value as long as $g_{t\phi} > \sqrt{-g_{\phi\phi}\, g_{tt}}$.

Lightcone slopes:

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The clear discriminator of the arrows of time are the slopes of the local light-cone,

$$s_\pm(r) = (r\dot{\phi}_\pm)^{-1} = \frac{1}{r}\,\frac{g_{\phi\phi}}{g_{t\phi} \pm \sqrt{-g_4}} = -\frac{1}{r}\,\frac{g_{t\phi} \mp \sqrt{-g}}{g_{tt}}\,. \tag{7}$$

Notice that if $g_{\phi\phi}$ and g_{tt} are positive, then regardless of the sign of $g_{t\phi}$, the light-cones (worldlines) remain in the first and second quadrants of the (t,ϕ) plane (as is the case of the Minkowski light-cone). Thus, for a backward flow of time, $g_{\phi\phi}$ (or, g_{tt}) must go through zero and become negative.

Product of lightcone slopes:

It is useful to consider the product of slopes

$$s_+(r)\,s_-(r) = \frac{-1}{r^2}\frac{g_{\phi\phi}}{g_{tt}}. \tag{8}$$

For time to move backwards one of the world lines defining the light-cone must move into the lower half of the $t-\phi$ plane. From (8) one can see that (i) this happens smoothly if $g_{\phi\phi}$ goes through zero; (ii) happpens discontinuously if g_{tt} goes through zero; (iii) that a smooth change in the sign of $g_{t\phi}$ cannot move either slope through zero to the domain of negative time.

With the focus here on a smooth change of sign for $g_{\phi\phi}$, it is useful to examine the slopes at small $g_{\phi\phi}$. One finds

$$s_\pm(\text{leading order in } g_{\phi\phi}) = \begin{cases} \frac{1}{2r}\frac{g_{\phi\phi}}{g_{t\phi}} \\ \\ -\frac{2}{r}\frac{g_{t\phi}}{g_{tt}} \end{cases} \tag{9}$$

It is clear that the slope s_+ goes through zero with $g_{\phi\phi}$, leaving the first quadrant and moving into the fourth quadrant. With increasing ϕ, time for the associated co-rotating world line runs backwards. On the other hand, the sign of s_- remains unchanged, and time for the associated counter-rotating world line continues to run forward. In the following we will apply similar arguments to different scenarios of asymmetrically warped spacetimes.

GTvS - the good, bad, and ugly

It is instructive to mention the visceral arguments against the relevance of the Gödel and TvS metrics. First of all, they are not asymptotically flat, and so presumably cannot occur within our Universe; rather, they must be our Universe, which contradicts observation. Secondly, the initial conditions from which they can evolve are either non-existent (Gödel) or sick (TvS). Furthermore, the TvS metric assumes an infinitely-long cylinder of matter, which is unphysical. On the positive side, literally, the Einstein equation endows $\rho = T^0_{\ 0} = (R^0_{\ 0} - \frac{1}{2}R)/8\pi G_N$ (with the geometric RHS determined by the metric) with a positive value everywhere; there is no need for "exotic" $\rho < 0$ matter. A further positive feature is the simplicity of finding the CTC by travel along the periodic variable ϕ.

A linear path off the brane

We may replace the periodic coordinate of GTvS with the unbounded x coordinate, and omit the y and z coordinates for brevity. Then one obtains

$$ds^2 = g_{tt}(u,v)\,dt^2 + 2g_{tx}(u,v)\,dxdt - g_{xx}(u,v)\,dx^2 - du^2 - dv^2\,. \tag{12}$$

(Notice in particular the sign convention on the coefficient of dx^2.)

The speed of light at any point will depend on (u,v) through the metric elements. The restriction to Lorentzian signature implies that

$$-g_6 \equiv -\mathrm{Det}(g_{\mu\nu}) = g_{tt}(u,v)\,g_{xx}(u,v) + g_{tx}^2(u,v) > 0\,. \tag{13}$$

World lines for lightlike travel (null lines) satisfy

$$0 = g_{tt}(u,v) + 2g_{tx}(u,v)\,\dot{x} - g_{xx}(u,v)\,\dot{x}^2 - \dot{u}^2 - \dot{v}^2\,. \tag{14}$$

The solutions to (14) for the analogs of co-rotating and counter-rotating light speed at fixed (u,v) are

$$\dot{x}_{\pm} = \frac{g_{tx}(u,v) \pm \sqrt{-g_6}}{g_{xx}(u,v)}\,. \tag{15}$$

On the brane, $\dot{x}$ must equal $c = 1$, so we again choose $g_{tt}(0,0) = g_{xx}(0,0) = 1$ and $g_{tx}(0,0) = 0$.

Causal properties:

Let us examine more closely the causal implications of Eq. (15). We assume that g_{tt} is everywhere positive, so that (i) coordinate time t is everywhere timelike, and (ii) no singularities are introduced in s_+s_- or in g_{tt}. As shown in Eqn. (8), the sign of g_{tx} does not influence the causal structure, and for definiteness we take it to be positive semidefinite. It is the sign of the metric element g_{xx} that has smooth causal significance.

Similar to the causal analysis of the GTvS model of Section (2), we write the two slopes of the light-cone as

$$s_\pm(u,v) = (\dot{x}_\pm)^{-1} = \frac{g_{xx}(u,v)}{g_{tx}(u,v) \pm \sqrt{g_{tx}^2(u,v) + g_{xx}(u,v)g_{tt}(u,v)}}. \tag{16}$$

From this, one readily gets

$$s_+s_- = \frac{-g_{xx}}{g_{tt}}. \tag{17}$$

It is easily seen that when g_{xx}, g_{tt}, and g_{tx} are all positive, the slopes are of opposite sign, and are connected to the Minkowski metric in the smooth limit $g_{tx} \to 0$. Thus, with g_{tt} and g_{tx} assumed positive, time flows in the usual manner if g_{xx} is positive. Furthermore, with $g_{xx} > 0$, we have $\mathrm{sign}(g_{tx} \pm \sqrt{-g_6}) = \pm$, so that from Eq. (15), one has $\dot{x}_+ > 0$, and $\dot{x}_- < 0$. Thus, a positive g_{xx} (as in the Lorentz metric) offers the standard situation with time flowing forward and velocity $\dot{x}$ having either sign.

Causal properties (continued) :

On the other hand, if g_{xx} is negative, then Eq. (17) shows that one light-cone slope has changed sign. The small g_{xx} limit of the slopes

$$s_{\pm}(\text{leading order in } g_{xx}) = \begin{cases} \frac{g_{xx}}{2g_{tx}} \\ \\ -\frac{2g_{tx}}{g_{tt}} \end{cases} \tag{18}$$

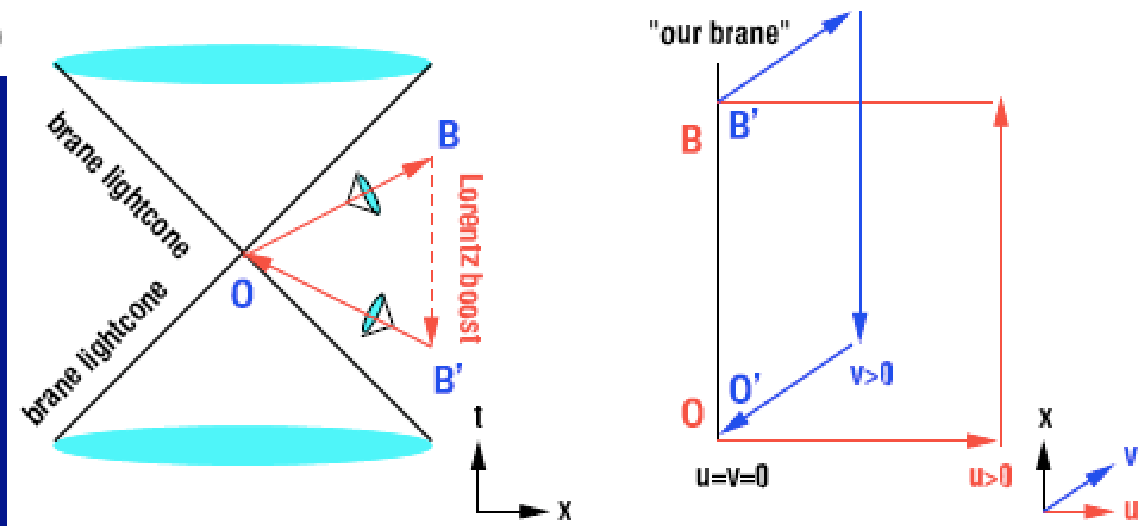
reveals that it is the positive slope which has passed through zero to become negative, signifying a world line moving from the first quadrant, through the x-axis, into the fourth quadrant where times flows backwards for increasing x. With both slopes negative, one has that $\dot{x}_{\pm}(g_{xx} < 0) < 0$. Thus, travel with increasing time is in the negative x direction, while travel with decreasing time is in the positive x direction. We summarize the causal properties of the metric (12) in Table 1.

	$g_{xx} > 0$	$g_{xx} < 0$
$\Delta T > 0$	$\dot{x}_+ > 0$ $(\Delta x > 0)$	
	$\dot{x}_- < 0$ $(\Delta x < 0)$	$\dot{x}_- < 0$ $(\Delta x < 0)$
$\Delta T < 0$		$\dot{x}_+ < 0$ $(\Delta x > 0)$

Table 1: Solution types for metric (12), and their casual properties. In particular, no solution exists for motion backwards in time along the negative-x direction.

CTC "construction":

The CTC which we investigate is the following: the signal travels first from the brane at $(u,v) = (0,0)$ to the hyperslice at (u_1,v_1), then from (u_1,v_1) to the hyperslice at (u_2,v_2), and finally back from (u_2,v_2) to the point of origin $(0,0)$ on the brane (see Fig. 1). While on the (u_1,v_1) hyperslice, the signal travels a distance ΔX in the positive x-direction over a negative time $\Delta T_1 = -|\Delta T_1|$. While on the (u_2,v_2) hyperslice, the signal travels back an equal negative distance $-\Delta X$ in time ΔT_2 to close the spatial projection of the worldline on the brane. To close the worldline on the brane, it is necessary that $T_2 + T_1 < 0$. (But not equal to zero, as we allow for small positive travel times from the brane at $(u,v) = (0,0)$ to (u_1,v_1), from (u_1,v_1) to (u_2,v_2), and back from (u_2,v_2) to $(0,0)$.)



Negative time:

The transit time $(\Delta T_1)_\pm$ for light to travel a positive distance $\Delta X > 0$ at constant (u_1, v_1), as viewed from the brane, is

$$(\Delta T_1)_\pm = \int_0^{\Delta T_1} dt = \int_0^{\Delta X} dx \, \frac{g_{xx}(u_1,v_1)}{g_{tx}(u_1,v_1) \pm \sqrt{-g_6(u_1,v_1)}} \qquad (19)$$

$$= \Delta X \left(\frac{g_{xx}(u_1,v_1)}{g_{tx}(u_1,v_1) \pm \sqrt{-g_6(u_1,v_1)}} \right).$$

The integrations on dt and dx are trivial because the metric does not depend on the coordinate time t or brane variable x. According to Eq. (13), the Lorentz signature is maintained as long as $g_{tx}^2 > g_{tt}(-g_{xx})$. We have shown that the world line for x_+ lies below the x-axis when $g_{xx} < 0$, and so we require $g_{xx}(u_1, v_1) < 0$ in order to gain negative time $\Delta(T_1)_+$ during travel on the (u_1, v_1) hyperslice. From here on, we will simply use the label ΔT_1 for this negative $\Delta(T_1)_+$ solution on the (u_1, v_1) hyperslice:

$$\Delta T_1 \equiv \Delta(T_1)_+ = \Delta X \left(\frac{g_{xx}(u_1,v_1)}{g_{tx}(u_1,v_1) + \sqrt{-g_6(u_1,v_1)}} \right). \qquad (20)$$

Return path:

To close the worldline, the lightlike signal must return from positive ΔX to the origin $x=0$ in a time ΔT_2 less than or equal to $|\Delta T_1|$. If this were to occur in a negative time, then we would have $g_{xx}<0$ and $\dot{x}>0$. Table 1 shows that there is no solution of this type available. So the return path must take place in positive time, with $\dot{x}<0$. Reference again to Table 1 reveals that the return solution is $\dot{x}_-$. In principle, the $\dot{x}_-$ solution on the (u_1,v_1) hyperslice provides a return path. However, it is easy to show that the return time ΔT_2 for this solution exceeds $|\Delta T_1|$ and so fails to close the world line. Thus, we must go to a second hyperslice at (u_2,v_2). We have

$$\Delta T_2 = \left[\int_{\Delta X}^{0} dx = -\Delta X\right]\left(\frac{g_{xx}(u_2,v_2)}{g_{tx}(u_2,v_2)-\sqrt{-g_6(u_2,v_2)}}\right), \qquad (21)$$

with $g_{xx}(u_2,v_2)$ of either sign.

CTC condition(s) :

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The necessary condition relating the outgoing and return paths of a CTC is that the sum $\Delta T_2 + \Delta T_1$ be less than zero. Equivalently, the CTC conditions are that

$$\frac{-g_{xx}(u_2,v_2)}{g_{tx}(u_2,v_2)-\sqrt{-g_6(u_2,v_2)}}+\frac{g_{xx}(u_1,v_1)}{g_{tx}(u_1,v_1)+\sqrt{-g_6(u_1,v_1)}}<0\,, \tag{22a}$$

and that

$$g_{xx}(u_1,v_1)<0\,, \tag{22b}$$

CTCs in N+1, N=2,3,....

We have seen GTvS CTCs in N=2 (phi,r).

Therefore, we expect metrics in N=3,4,... with CTCs.

Eqn (22) shows that this is so.

The mathematical recipe that emerges from Eqn (22) is simply:

(i) allow g_{xx} to change sign as a function of another spatial variable;

(ii) take g_{tx} nonzero;

(iii) arrange a suitably “fast” return path.

AND, there is an aesthetic input:

choose a metric that is physically motivated.

Tom Weiler, Vanderbilt University, USA

Causality with one warped dimension:

We consider the five-dimensional asymmetrically-warped line element with a single extra dimension which we label as "u":

$$ds^2 = dt^2 - \sum_i \alpha^2(u)\,(dx^i)^2 - du^2, \tag{24}$$

$i = 1,2,3$, with our brane located at the $u = 0$ submanifold. With no loss of generality, we may take $\alpha(u)$ to be positive.

Boosting, $$ds'^2 = \gamma^2 \left\{ \left(1-\beta^2\alpha_2^2\right) dt'^2 + 2\beta\left(1-\alpha_2^2\right) dx'dt' - \left(\alpha_2^2 - \beta^2\right) dx'^2 \right\}. \tag{32}$$

This looks promising, of the GTvS form.

The warped spacetime of (24) allows shortcut geodesics connecting spacelike-separated events on the brane if $|\alpha(u)| < |\alpha(0)|$ for any $u \neq 0$. However, the metric (24) exhibits a global time function t. Thus, taken by itself this spacetime is causally stable and does not allow for CTCs.

The failure of (24) to support a CTC can also be seen in our CTC equations (22). Since $g_{xx} = \alpha^2$ in (24) cannot be negative without violating the assumed Lorentzian signature, the CTC condition (22b) cannot be satisfied. This failure can be traced to the fact that the Lorentz transformation was just a coordinate change, and so provided a change of view, but no new physics. What is needed is a nonzero g_{tx} that cannot be removed by a linear transformation among brane coordinates. Introducing the 6th dimension provides a solution, first because it allows a superluminal return path along the additional 6th dimension, and second because it allows $g_{tx}(u,v)$ to be "hard-wired" into the metric so that it is not removable by a linear coordinate transformation on the brane.

CTCs with two warped extra dimensions:

A natural 6D generalization of (24) can be realized by assuming that the metric for the u- and v dimensions exhibits the simple form in (24), but in different Lorentz frames. This assumption seems natural for any spacetime with two or more extra dimensions, since there is no preferred Lorentz frame for the bulk, from the viewpoint of the brane.

This choice also ensures superluminal travel to as well as from the brane, as well as a Minkowskian metric on the brane. To construct this 6-dimensional metric explicitly, let us denote by β_{uv} the "relative velocity" between the two Lorentz frames in which the u and v dimensions assume the simple form (24), respectively. We incorporate the "u-frame" slice at $v=0$ by retaining the warp factor $\alpha(u)$ on the brane coordinate dx, and we incorporate the "v-frame" slice at $u=0$ by writing the boosted metric in Eq. (32) with the warp $\alpha(u)$ now replaced by $\eta(v)$. The resulting full 6-dimensional metric then has the form

$$ds^2 = \gamma_{uv}^2 \left\{ [1-\beta_{uv}^2\eta^2(v)]\, dt^2 + 2\beta_{uv}\alpha(u)[1-\eta^2(v)]\, dx dt - \alpha^2(u)[\eta^2(v)-\beta_{uv}^2]\, dx^2 \right\} - du^2 - dv^2. \tag{34}$$

One easily finds that $-Det \equiv -g_6 = \alpha^2(u)\eta^2(v)$. That this determinant is independent of β_{uv} is consistent with the interpretation of β_{uv} as a kind of boost parameter. Of special importance for the existence of the CTC is the off-diagonal metric element g_{tx}, which is nonzero for $\eta(v) \neq 1$ (i.e., off the brane), and the metric element g_{xx} which is of indeterminate sign.

One simple and successful choice is to set $\alpha_1 = 1$ and $\eta_2 = 1$, i.e., to take the outgoing path on the $u = 0$ hyperslice and the return path on the $v = 0$ hyperslice. With these choices, (37) reduces to $\alpha_2 < (\beta - \eta_1)/(1 - \beta\eta_1)$. This is guaranteed to be satisfiable by (36).

With $u = 0$, Eq. (34) reduces to (32) with $\eta^2(v)$ replacing $\alpha^2(u)$:

$$ds^2|_{u=0} = \gamma_{uv}^2 \{ [1 - \beta_{uv}^2 \eta^2(v)]dt^2 + 2\beta_{uv}[1 - \eta^2(v)]dxdt - [\eta^2(v) - \beta_{uv}^2]dx^2 \} - du^2 - dv^2. \quad (38)$$

Thus we see explicitly that choosing $\eta_1 < \beta_{uv}$ on the $u = 0$ hyperslice sets $g_{xx} < 0$, so that our outgoing path necessarily accumulates negative time (original frame in Table 2). On the return path, we set $v = 0$. Then the 6D metric of Eq. (34) reduces to (24), repeated here:

$$ds^2|_{v=0} = dt^2 - \alpha^2(u)dx^2 - du^2; \quad (39)$$

It is clear that this return path can be made arbitrarily brief by choosing α_2 arbitrarily small. The CTC is revealed.

Stress-energy tensor and energy conditions

As a check on the consistency of the picture, we should diagnose the stress-energy tensor which sources the extra-dimensional metric, for any pathologies. In particular, we will be interested in the resulting matter distributions on and off the brane. Thus, our task is to calculate the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{40}$$

from the spacetime metric of Eq. (34), and then to obtain the stress-energy tensor $T_{\mu\nu}$ via the Einstein equation

$$T_{\mu\nu} = \frac{1}{8\pi G_N} G_{\mu\nu}. \tag{41}$$

We note that in general, $T_{\mu\nu}$ contains contributions from matter, fields, and cosmological constant on and off the brane, and from brane tension on the brane.

Then, with a diagonal metric with $g_{tt} = 1$ (Gaussian-normal coordinates), one obtains for the nonzero elements of T^{μ}_{μ},

$$\rho = T^0_0 \quad \text{and} \quad p^j = -T^j_j. \tag{48}$$

These are the relations appropriate for the $v = 0$ slice of our metric, since one sees in Eq. (39) that the $v = 0$ metric is manifestly diagonal with $g_{tt} = 1$.

It is not difficult to find a functional form for the warp factors α and η which conserves some of the energy conditions, at least on the brane. One such example is given by $\alpha(u) = 1/(u^2 + c^2)$ and $\eta(v) = 1/(v^4 + c^2)$. For this case the elements of the Einstein tensor on the $v = 0$ slice are shown as a function of u in Fig. 2. The null, weak and dominant energy conditions are conserved on the brane, while the strong energy condition is violated both on the brane and in the bulk.

A good discussion/display of various energy-conditions in S. Carroll's GR book.

Energy figure

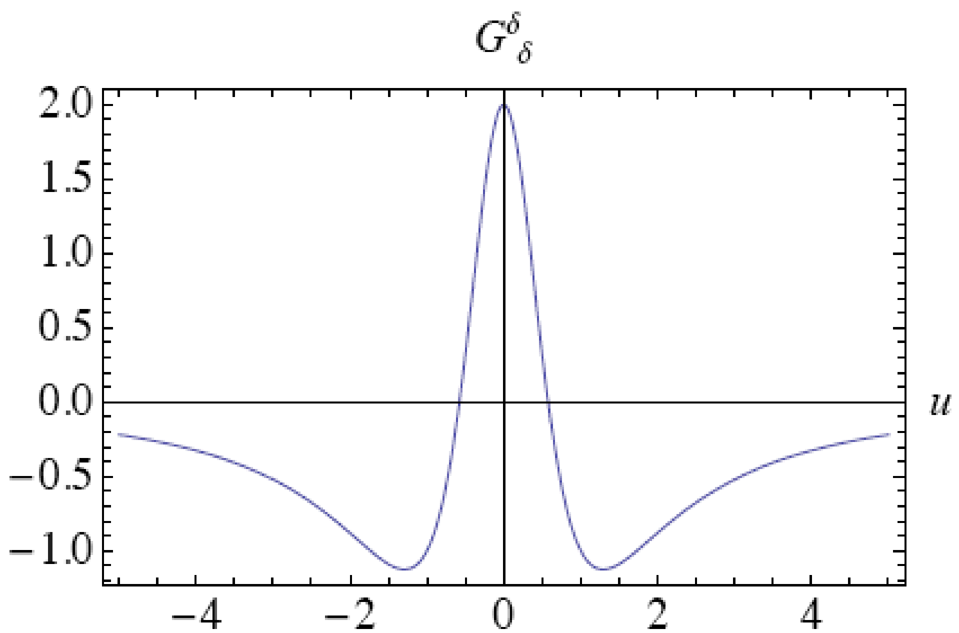

Figure 2: Nonzero elements of the Einstein tensor $G^{\mu}{}_{\nu}$ (in arbitrary units): $G^{\delta}{}_{\delta} \equiv G^{0}{}_{0} = G^{y}{}_{y} = G^{z}{}_{z} = G^{v}{}_{v}$, on the $v = 0$ slice, as a function of u. Assumed are warp factors $\alpha(u) = 1/(u^2 + c^2)$ and $\eta(v) = 1/(v^4 + c^2)$, with $c = 1$. We find that the weak and dominant energy conditions are violated in the bulk, while all energy conditions with the exception of the SEC are satisfied on the brane.

Energy (philosophic discussion)

The negative energy density that afflicts many wormhole and CTC solutions in four dimensions is avoided on the brane in the example for an extra-dimensional CTC presented here. However, ρ becomes negative as one moves away from the brane into the bulk, so that the WEC and DEC are violated off the brane, while the NEC remains conserved. We have successfully constructed a metric exhibiting CTCs in an extra-dimensional spacetime by "moving" the negative energy density from the brane to the bulk. One might even speculate that the negative energy density in the bulk is related to the compactification of the extra dimensions, or possibly to the repulsion of Standard matter from the bulk.

One also sees in Fig. (2) that $G^y{}_y = G^z{}_z = G^v{}_v$ are equal to $G^0{}_0$ on the $v = 0$ slice. This equality amounts to a dark energy or cosmological constant equation of state for the y-, t-, and v-directed pressures, namely, $w^j \equiv p^j/\rho = -1$. There may be some intriguing physics underlying this result.

It is also possible that an anthropic argument applies here: Life may evolve only where energy density is positive. Then lifeless bulk regions of negative energy density can communicate their existence to living beings only via geometry, perhaps mediated by the exchange of gravitons or appropriately named, "sterile" neutrinos.

Summary

* GTvS CTCs easily generalize to more dimns
  (general CTC conditions on metric given)
* There exist spatially warped metrics in 6D
  (not 5D) exhibiting CTCs
* These CTCs challenge “chronology protection”,
  and may enable iner-temporal communication
* Intriguing energetics,
  positive on brane,
  negative in bulk
* More implications, more models to investigate

Extra slides

Energy conditions:

There is considerable theoretical prejudice that stable Einstein tensors should satisfy certain "energy conditions" relating energy density ρ and directional pressures p^j. The null, weak, strong and dominant energy conditions state that

$$\text{NEC} \quad : \quad \rho + p^j \geq 0, \quad \forall j. \tag{42}$$

$$\text{WEC} \quad : \quad \rho \geq 0; \quad \text{and} \quad \rho + p^j \geq 0, \quad \forall j. \tag{43}$$

$$\text{SEC} \quad : \quad \rho + p^j \geq 0, \quad \forall j; \quad \text{and} \quad \rho + \sum_j p^j \geq 0. \tag{44}$$

$$\text{DEC} \quad : \quad \rho \geq 0; \quad \text{and} \quad p^j \in [\rho, -\rho], \quad \forall j. \tag{45}$$

Tom Weiler, Vanderbilt University, USA