Model building and Lie point symmetries

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Overview

Want to systematically find all the symmetries of an action,

$\rightarrow$ even if symmetry is spontaneously broken,

$\rightarrow$ also derive parameter relationships that give enhanced symmetries.

Overview:

- The Lie point symmetry (LPS) method.
- Example with 2 scalars.
- N interacting scalars.
- Spontaneous symmetry breaking.
- Automation of large systems.
- A catalogue of field theories.
- The standard model.

The Lie point symmetry method

The Lie point symmetry method consists of finding the determining equations, whose solutions describe infinitesimal symmetries, and then solving these equations.

Point: transformations depend only on the coordinates and fields themselves, and not the derivatives of the fields.

1. Derive the determining equations of the system.
   x^μ and fields ϕ_i, one makes a general infinitesimal variation of the coordinates, $\delta x^\mu = \eta^\mu$, and fields, $\delta\phi_i = \chi_i$,
2. Solve the determining equations, or at least reduce them to a standard form. Here we can obtain branching of the solution depending on parameter values.
3. Compute the rank of the symmetry set(s).
   $R = (N_{\text{const}}, N_{f_1}, N_{f_2}\ldots)$.
4. Compute the action of the symmetries.

The method

The LPS method is a very general and powerful tool:

- it is an exhaustive search of continuous symmetries;
- it yields all interesting relationships between parameters;
- finding the rank is guaranteed to terminate in finite time, determined by the number of coordinates and number of fields.

Variation of the action

Lie point symmetries: $\eta^\mu(x,\phi)$, $\chi_i(x,\phi)$.

$$x^\mu \to x^\mu + \eta^\mu \qquad\qquad S \to S + \delta S \text{ should be unchanged.}$$
$$\phi_i \to \phi_i + \chi_i$$

Solve for the fields $\to$ Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) = 0\,.$$

Solve for the infinitesimals $\to$ master determining equation:

$$\mathcal{L}\frac{\mathrm{d}\eta^\mu}{\mathrm{d}x^\mu} + \frac{\partial \mathcal{L}}{\partial x^\mu}\eta^\mu + \frac{\partial \mathcal{L}}{\partial \phi_i}\chi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \left(\frac{\mathrm{d}\chi_i}{\mathrm{d}x^\mu} - \frac{\partial \phi_i}{\partial x^\nu}\frac{\mathrm{d}\eta^\nu}{\mathrm{d}x^\mu} \right) = 0\,,$$

where

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu} \equiv \frac{\partial}{\partial x^\mu} + \frac{\partial \phi_i}{\partial x^\mu}\frac{\partial}{\partial \phi_i}\,.$$

Example: two scalars

No coordinates symmetries, $\phi_i \to \phi_i + \chi_i(\phi_i)$.
Master determining equation:

$$\frac{\partial \mathcal{L}}{\partial \phi_i}\chi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\frac{\partial \phi_j}{\partial x^\mu}\frac{\partial \chi_i}{\partial \phi_j} = 0\,.$$

Apply to Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi_1 \partial_\mu \phi_1 + \frac{1}{2}\partial^\mu \phi_2 \partial_\mu \phi_2 - \frac{1}{2}m_1^2\phi_1^2 - \frac{1}{2}m_2^2\phi_2^2\,.$$

Determining equation is

$$\begin{aligned} &- m_1^2\phi_1\chi_1 - m_2^2\phi_2\chi_2 + \partial^\mu\phi_1\partial_\mu\phi_1\frac{\partial \chi_1}{\partial \phi_1} \\ &\quad + \partial^\mu\phi_1\partial_\mu\phi_2\frac{\partial \chi_1}{\partial \phi_2} + \partial^\mu\phi_2\partial_\mu\phi_1\frac{\partial \chi_2}{\partial \phi_1} + \partial^\mu\phi_2\partial_\mu\phi_2\frac{\partial \chi_2}{\partial \phi_2} = 0\,. \end{aligned}$$

Equate independent terms to zero:

$$-m_1^2\phi_1\chi_1 - m_2^2\phi_2\chi_2 = 0\,, \quad \frac{\partial \chi_1}{\partial \phi_1} = 0\,, \quad \frac{\partial \chi_1}{\partial \phi_2} + \frac{\partial \chi_2}{\partial \phi_1} = 0\,, \quad \frac{\partial \chi_2}{\partial \phi_2} = 0\,.$$

Example: two scalars

Determining equations:

$$-m_1^2\phi_1\chi_1 - m_2^2\phi_2\chi_2 = 0\,, \quad \frac{\partial\chi_1}{\partial\phi_1} = 0\,, \quad \frac{\partial\chi_1}{\partial\phi_2} + \frac{\partial\chi_2}{\partial\phi_1} = 0\,, \quad \frac{\partial\chi_2}{\partial\phi_2} = 0\,.$$

General solution to last three equations:

$$\chi_1(\phi_2) = \alpha_1 + \beta\phi_2\,, \qquad \chi_2(\phi_1) = \alpha_2 - \beta\phi_1\,.$$

Maximum rank $R = 3$. Symmetries:

- α_1: shift of ϕ_1.
- α_2: shift of ϕ_2.
- β: rotation between ϕ_1 and ϕ_2.

Final determining equation is

$$\alpha_1 m_1^2\phi_1 + \alpha_2 m_2^2\phi_2 + \beta(m_1^2 - m_2^2)\phi_1\phi_2 = 0\,.$$

$\longrightarrow$ *the model parameters dictate the symmetries.*

Example: two scalars

Recall the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi_1 \partial_\mu \phi_1 + \frac{1}{2}\partial^\mu \phi_2 \partial_\mu \phi_2 - \frac{1}{2}m_1^2\phi_1^2 - \frac{1}{2}m_2^2\phi_2^2\,.$$

General solution for symmetries

$$\phi_1 \to \bar{\phi}_1 \quad \text{with} \quad \bar{\phi}_1' = \chi_1 = \alpha_1 + \beta\bar{\phi}_2$$
$$\phi_2 \to \bar{\phi}_2 \quad \text{with} \quad \bar{\phi}_2' = \chi_2 = \alpha_2 - \beta\bar{\phi}_1$$

Final algebraic determining equation

$$\alpha_1 m_1^2 \phi_1 + \alpha_2 m_2^2 \phi_2 + \beta(m_1^2 - m_2^2)\phi_1\phi_2 = 0\,.$$

- $m_1 = 0$ allows $\alpha_1 \neq 0$, symmetry $\bar{\phi}_1 = \phi_1 + \alpha_1\epsilon$.
- $m_2 = 0$ allows $\alpha_2 \neq 0$, symmetry $\bar{\phi}_2 = \phi_2 + \alpha_2\epsilon$.
- $m_1^2 = m_2^2$ allows $\beta \neq 0$. The symmetry is

$$\begin{pmatrix}\bar{\phi}_1 \\ \bar{\phi}_2\end{pmatrix} = \begin{pmatrix}\cos\beta\epsilon & \sin\beta\epsilon \\ -\sin\beta\epsilon & \cos\beta\epsilon\end{pmatrix}\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}\,.$$

N interacting scalar fields

Symmetries dictated by structure of interactions between fields.

General Lagrangian for N spin-0 fields

$$\mathcal{L} = T_{ij}\partial^\mu \phi_i \partial_\mu \phi_j - V(\phi)\,.$$

Determining equations are

$$\begin{aligned}
V\partial_\mu \eta^\mu + \frac{\partial V}{\partial \phi_i}\chi_i &= 0\,, \\
\partial^\mu \chi_i - V\frac{\partial \eta^\mu}{\partial \phi_i} &= 0 \qquad \forall \mu\ \forall i\,, \\
\partial^\mu \eta^\nu + \partial^\nu \eta^\mu &= 0 \qquad \forall \mu\ \forall \nu,\ \mu \neq \nu\,, \\
\frac{\partial \chi_i}{\partial \phi_j} + \frac{\partial \chi_j}{\partial \phi_i} &= 0 \qquad \forall i\ \forall j,\ i \neq j\,, \\
\frac{1}{2}\partial_\sigma \eta^\sigma - \partial_{\bar{\mu}} \eta^{\bar{\mu}} + \frac{\partial \chi_{\bar{i}}}{\partial \phi_{\bar{i}}} &= 0 \qquad \forall \bar{\mu}\ \forall \bar{i}\,, \\
\frac{\partial \eta^\mu}{\partial \phi_i} &= 0 \qquad \forall \mu\ \forall i\,.
\end{aligned}$$

N interacting scalar fields

Recall general Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi_i \partial_\mu \phi_i - V(\phi) \,.$$

For $D \neq 2$ the general coordinate symmetries are ($b^{\mu\nu}$ anti-symm)

$$\eta^\mu(x) = a^\mu + b^\mu{}_\nu x^\nu - \frac{2\gamma}{D-2} x^\mu \,.$$

General field symmetries are (β_{ij} anti-symm)

$$\chi_i(\phi) = \alpha_i + \beta_{ij}\phi_j + \gamma\phi_i \,.$$

Remaining determining equation is $[d \equiv 2D/(D-2)]$

$$-d\gamma V + \frac{\partial V}{\partial \phi_i}\left(\alpha_i + \beta_{ij}\phi_j + \gamma\phi_i\right) = 0 \,.$$

Form of V $\leftrightarrow$ allowed symmetries.

Symmetries of one scalar

Specialise to $N = 1$:

$$-d\gamma V + \frac{\mathrm{d}V}{\mathrm{d}\phi}(\alpha + \gamma\phi) = 0\,.$$

Four distinct cases:

$V = 0$: α and γ free. Independent shift and scale symmetries.
Rank associated with field is $R_\chi = (2)$.

$V = \text{const}$: $\gamma = 0$ but α is free.
Field rank $R_\chi = (1)$.

$V = \lambda(\phi + v)^d$: Solve above differential equation.
Given v, relationship between shift and scale symmetry is fixed by $v = \alpha/\gamma$.
Field rank $R_\chi = (1)$.

V arbitrary: $\alpha = \gamma = 0$. No shift or scale symmetry.
Field rank $R_\chi = (0)$.

Symmetries of two scalars

$$-d\gamma V + \frac{\partial V}{\partial \phi_1}(\alpha_1 + \beta\phi_2 + \gamma\phi_1) + \frac{\partial V}{\partial \phi_2}(\alpha_2 - \beta\phi_1 + \gamma\phi_2) = 0\,.$$

Go to polar field variables, $\phi_1 = r\cos\theta$, $\phi_2 = r\sin\theta$:

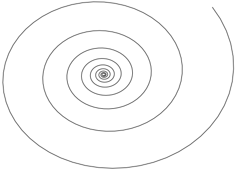
$$\mathcal{L} = \frac{1}{2}\partial^\mu r \partial_\mu r + r^2 \frac{1}{2}\partial^\mu\theta\partial_\mu\theta - V(r,\theta)\,.$$

Determining equation is

$$-d\gamma V + \frac{\partial V}{\partial r}(\alpha_1\cos\theta + \alpha_2\sin\theta + \gamma r) - \frac{\partial V}{\partial \theta}\left(\alpha_1\frac{\sin\theta}{r} - \alpha_2\frac{\cos\theta}{r} + \beta\right) = 0.$$

A solution:

$$V(r,\theta) = \lambda\left(r^k - v\mathrm{e}^{l\theta}\right)^m\,.$$



k and m related by $mk = d$. Relationship between scale and rotation symmetry fixed by $k\gamma = l\beta$. Action of the symmetry is $r \to \mathrm{e}^\gamma r$, $\theta \to \theta - k\gamma/l$ and $x^\mu \to \mathrm{e}^{-d\gamma/D}x^\mu$.

The action versus the equations of motion

Distinction between the symmetries of action and symmetries of corresponding equations of motion.

G a symmetry of an action $\implies$ G also a symmetry of the Euler-Lagrange equations. Converse not necessarily true.

Denote the system by $\Delta_j(x^\mu, \phi_i, \partial\phi_i) = 0$.

1. Construct the prolonged symmetry operator $\mathrm{pr}^{(k)}\,\boldsymbol{\alpha}$.

$$\boldsymbol{\alpha} = \eta^\mu \frac{\partial}{\partial x^\mu} + \chi_i \frac{\partial}{\partial \phi_i}\;.$$

   Prolongation extends $\boldsymbol{\alpha}$ to include all possible combinations of derivatives of ϕ, to order k.

2. Apply $\mathrm{pr}^{(k)}\,\boldsymbol{\alpha}$ to the system: $(\mathrm{pr}^{(k)}\,\boldsymbol{\alpha}\cdot\Delta)|_{\Delta=0} = 0$.
3. Equate all independent coefficients to zero $\rightarrow$ determining equations.

Equations of motion example

System defined by Euler-Lagrange equation $\ddot{\phi} - \phi'' + m^2\phi = 0$.

What are its symmetries?

- $m = 0$ has

$$\eta^t(t,x) = F_+(t+x) + F_-(t-x)\,,$$
$$\eta^x(t,x) = F_+(t+x) - F_-(t-x) + f\,,$$
$$\chi(t,x,\phi) = G_+(t+x) + G_-(t-x) + g\,\phi(t,x)\,.$$

- $m \neq 0$ has

$$\eta^t(x) = a^t + bx\,,$$
$$\eta^x(t) = a^x + bt\,,$$
$$\chi(t,x,\phi) = \int_{-\infty}^{+\infty} dk \left[H_+(k)\,\mathrm{e}^{i(\omega t + kx)} + H_-(k)\,\mathrm{e}^{i(\omega t - kx)} \right] + g\,\phi(t,x)\,,$$

where $\omega = \sqrt{k^2 + m^2}$.

Remarks on the LPS method

The LPS method provides an exhaustive list of symmetries and relationships between parameters such that an enhanced symmetry is obtained.

In the examples so far we have only considered spin-0 particles.
But, any spin representation, or even particles that do not respect Lorentz symmetry, can be written in terms of real fields.
Any action can be expanded in terms of its real components.

The LPS method works for all continuous symmetries that depend on the coordinates and fields (but not derivatives of the fields).
This includes local gauge symmetries as well as general relativity.
Supersymmetry also possible; requires the introduction of anti-commuting coordinates.

Works for non-linear symmetries and spontaneously broken symmetries.

Non-linear symmetries

Field (no coordinate) symmetries of

$$\mathcal{L} = \phi^m (\partial^\mu \phi \partial_\mu \phi)^n \,.$$

m and $n \neq 0$ are constant exponents.

Determining equation

$$m\phi^{m-1}\chi + 2n\phi^m \frac{\mathrm{d}\chi}{\mathrm{d}\phi} = 0 \,.$$

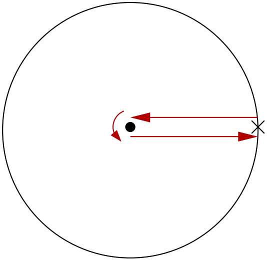
Solve for χ:

$$\chi = a\phi^{-m/2n} \qquad a \text{ is integration constant} \,.$$

Non-linear symmetry acts by $\bar{\phi}' = a\bar{\phi}^{-m/2n}$, solution

$$\phi \to (\phi^p + pa\epsilon)^{1/p} \qquad \text{with} \quad p = 1 + m/2n \,.$$

Spontaneously broken scale symmetry:
$V = \lambda(\phi + v)^d$ has shift-scale-shift symmetry.



$V = \lambda(\phi_1^2 + \phi_2^2 - v^2)^2$ has $\mathrm{U}(1)$.

Define $\phi_2 = v + \varphi$.

$V = \lambda(\phi_1^2 + \varphi^2 + 2v\varphi)^2$ has shift-$\mathrm{U}(1)$-shift.

LPS method: solve for c_i in

$$V = c_1 + c_2\phi_1 + c_3\phi_2 + c_4\phi_1^2 + c_5\phi_1\phi_2 + c_6\phi_2^2 + c_7\phi_1^3 + c_8\phi_1^2\phi_2 + c_9\phi_1\phi_2^2 + c_{10}\phi_2^3 + c_{11}\phi_1^4 + c_{12}\phi_1^3\phi_2 + c_{13}\phi_1^2\phi_2^2 + c_{14}\phi_1\phi_2^3 + c_{15}\phi_2^4$$

Find relationships between c_i and the symmetry rank.

Automation of LPS method

Large systems lead to an unmanageable set of determining equations.

The LPS method can be cast as a well defined algorithm that completes in finite time, at least up to finding parameter relationships and the rank of the symmetry.

We can construct a computer program which takes in a Lagrangian and returns a list of branches of symmetries and parameter relationships.

1. Compute the determining equations.
   Straightforward.
2. Reduce the determining equations to standard form.
   Difficult. Includes branching.
3. Compute rank of each branch.
   Simple.

Automation of LPS method

Two scalars again

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi_1\partial_\mu\phi_1 + \frac{1}{2}\partial^\mu\phi_2\partial_\mu\phi_2 - \frac{1}{2}m_1^2\phi_1^2 - \frac{1}{2}m_2^2\phi_2^2\,.$$

Algebraic determining equation

$$\alpha_1 m_1^2\phi_1 + \alpha_2 m_2^2\phi_2 + \beta(m_1^2 - m_2^2)\phi_1\phi_2 = 0\,.$$

Find null space of

$$\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_1^2 - m_2^2 \end{pmatrix}\begin{pmatrix}\alpha_1 \\ \alpha_2 \\ \beta\end{pmatrix} = 0\,.$$

$\longrightarrow$ use Gaussian elimination with branching.

Automation of LPS method

Differential equations $\rightarrow$ generalised Gaussian elimination.

Define ordering on η^μ and χ_i. Sort terms. Arrange as rows.

Perform "row reduction" to "diagonal" form.

Schematically:

$$c_1\,\partial_i f + X_1(f) = 0\,,$$
$$c_2\,\partial_{i+j} f + X_2(f) = 0\,.$$

Branch:

- $c_1 = 0$: remove $\partial_i f$ term.
- $c_1 \neq 0$: use $\partial_i f$ to eliminate $\partial_{i+j} f$.

Catalogue of field theories

Assume we have a program which takes $\mathcal{L}$ and returns all possible branches with parameter constraints and symmetry rank.

Since any model can be written in terms of its real components, we can start to make a comprehensive catalogue of all possible theories.

- Ordered on the number of real degrees of freedom N.
- Just write down the most general action with N fields (all possibly derivatives and couplings).
- Find, for example, relationships that give spin-0, spin-$\frac{1}{2}$ and so on.
- Large rank means highly symmetric model.
- Any model (up to certain N) will be in the catalogue.

Catalogue of field theories

Consider $N = 1$ in 2d:

$$\mathcal{S} = \int \mathrm{d}t\, \mathrm{d}x \Big[c_0 + c_1\phi + c_2\dot{\phi} + c_3\phi' + c_4\phi^2 + c_5\phi\dot{\phi} + c_6\phi\phi' + c_7\dot{\phi}^2 + c_8\dot{\phi}\phi' + c_9\phi'^2 \Big] .$$

The case $c_0 = c_1 = c_2 = c_3 = c_5 = c_6 = c_8 = 0$, $c_7 = -c_9$ is a massive spin-0 field in 2d.

Main disadvantage: many terms.

For 4d with renormalisable terms we obtain $\sim 26N^4$ terms for large N.

$N = 4$ has about 10^4 terms, allowing for a gauge field or a Weyl fermion.

For $N = 10$ there are about 3×10^5 terms.

Can reduce number of terms by analysing models with certain Lorentz structure.

The standard model

Apply the LPS method to the standard model:

- Prove that there are no hidden symmetries.
- Rediscover Higgs mechanism.
- Add new degrees of freedom and look for new symmetries.

Schematic structure of the standard model:

$$\mathcal{L}_{\mathsf{SM}} \sim (\partial\phi)^2 + \phi^2\partial\phi + \phi^2 + \phi^4 + \psi\partial\psi + \phi\psi^2 \,.$$

$N = 244$ real degrees of freedom (with RH neutrinos):

- gauge = 4 real components × (1 hyp + 3 weak + 8 strong) = 48,
- leptons = 8 real components × 3 gens × (ν + e) = 48,
- quarks = 8 real components × 3 gens × 3 cols × (u + d) = 144,
- and Higgs = 2 real components × weak-doublet = 4.

Handling the size of the standard model

Standard model is huge: about 10^7 terms.

Maximum number of determining equations: 2.5×10^6
(but many are duplicated, and many are single term).

To handle the standard model:

- Generic analysis by hand in index notation.
- Look at restricted set of fields (only leptons, only 1 generation).
- Use known values of parameters within experimental uncertainty.
- Linearise parameters around measure value.

Also have running of parameters.

The Higgs mechanism and beyond

The LPS method can handle spontaneously broken symmetries.

In the broken phase without Higgs degrees of freedom, the standard model has $\mathrm{SU}(3)_C$ and $\mathrm{U}(1)_Q$ gauge symmetries.

Adding four fields with specific couplings gives the standard model a shift-$\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$-shift symmetry.

The LPS method should be able to rediscover the Higgs mechanism in this way.

Add fields beyond the Higgs. Any other symmetries?

Beyond Lie point symmetries

- Contact symmetries: depend on first derivative of the field.
- Generalised symmetries (Lie-Bäcklund), which allow η^μ and χ_i to depend on arbitrary derivatives of ϕ_i.
- Discrete symmetries (which are not subsets of a continuous group). Hydon, Eur. J. of Appl. Math., 11 (2000) 515. $\rightarrow$ a method to systematically find discrete point symmetries.

Conclusions

Coordinate variation η^μ, field variation χ_i.
Master determining equation:

$$\mathcal{L}\frac{\mathrm{d}\eta^\mu}{\mathrm{d}x^\mu} + \frac{\partial\mathcal{L}}{\partial x^\mu}\eta^\mu + \frac{\partial\mathcal{L}}{\partial\phi_i}\chi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\left(\frac{\mathrm{d}\chi_i}{\mathrm{d}x^\mu} - \frac{\partial\phi_i}{\partial x^\nu}\frac{\mathrm{d}\eta^\nu}{\mathrm{d}x^\mu}\right) = 0$$

The Lie point symmetry method:

- Counterpart to the Euler-Lagrange equations.
- Finds all possible symmetries.
- Finds all interesting relationships between parameters.
- Works even for spontaneously broken symmetries.
- Can be automated; crucial for large systems.

Future work:

- Show that the Higgs mechanism can be rediscovered.
- Find all symmetries of the standard model.
- Add new fields to the standard model to see if new symmetries can be found.

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