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Anomaly-free scale symmetry and gravity

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Based on

M.S., Anna Tokareva: Exact quantum conformal symmetry, its spontaneous breakdown, and gravitational Weyl anomaly, Phys. Rev. D 107 (2023) 6; Anomaly-free scale symmetry and gravity, Phys. Lett. B 840 (2023) 137898

with inputs from Georgios Karananas, Vladimir Kazakov, MS: Spontaneous Conformal Symmetry Breaking in Fishnet CFT, Phys. Lett. B 811 (2020) 135922

and previous works together with Armillis, Bezrukov, Blas, Garcia-Bellido, Karananas, Monin, Mooij, Rubio, Shkerin, Shimada, Tkachev, Tkachov, Voumard, Zell, and Zenhausern

Based on

Many ideas coming from Christof Wetterich works on scale symmetry:

Fine Tuning Problem and the Renormalization Group, Phys. Lett. B 140 (1984) 215

Cosmology and the Fate of Dilatation Symmetry, Nucl. Phys. B 302 (1988) 668

Fundamental scale invariance, Nucl. Phys. B 964 (2021) 115326

Outline

- Conformal symmetry
- Spontaneously broken conformal symmetry
- Conformal symmetry and gravity: Weyl anomaly
- Anomaly free scale symmetry
- Scale symmetry and phenomenology
- Conclusions

Maxwell equations as a source for symmetries

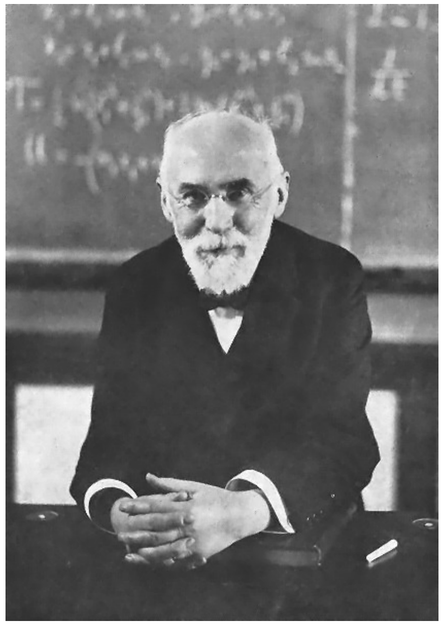
The study of free Maxwell equations:

$$\partial_\mu F^{\mu\nu} = 0, \qquad \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

led to the discovery of profound symmetries of space-time

Lorentz and Poincaré transformations, Minkowski space-time



translation

Lorentz transformation

$$x^\mu \rightarrow x'^\mu = x^\mu + x_0^\mu + \Lambda^\mu{}_\nu x^\nu$$

The group with 10 parameters: 4 translations, 3 rotations, 3 boosts

Conformal transformations

70 MR. H. BATEMAN [Nov. 12,

THE CONFORMAL TRANSFORMATIONS OF A SPACE OF FOUR DIMENSIONS AND THEIR APPLICATIONS TO GEOMETRICAL OPTICS

By H. BATEMAN.

[Received October 9th, 1908.—Read November 12th, 1908.]

1909.] THE PRINCIPLE OF RELATIVITY IN ELECTRODYNAMICS. 77

THE PRINCIPLE OF RELATIVITY IN ELECTRODYNAMICS AND AN EXTENSION THEREOF

By E. CUNNINGHAM.

[Received May 1st, 1909.]*

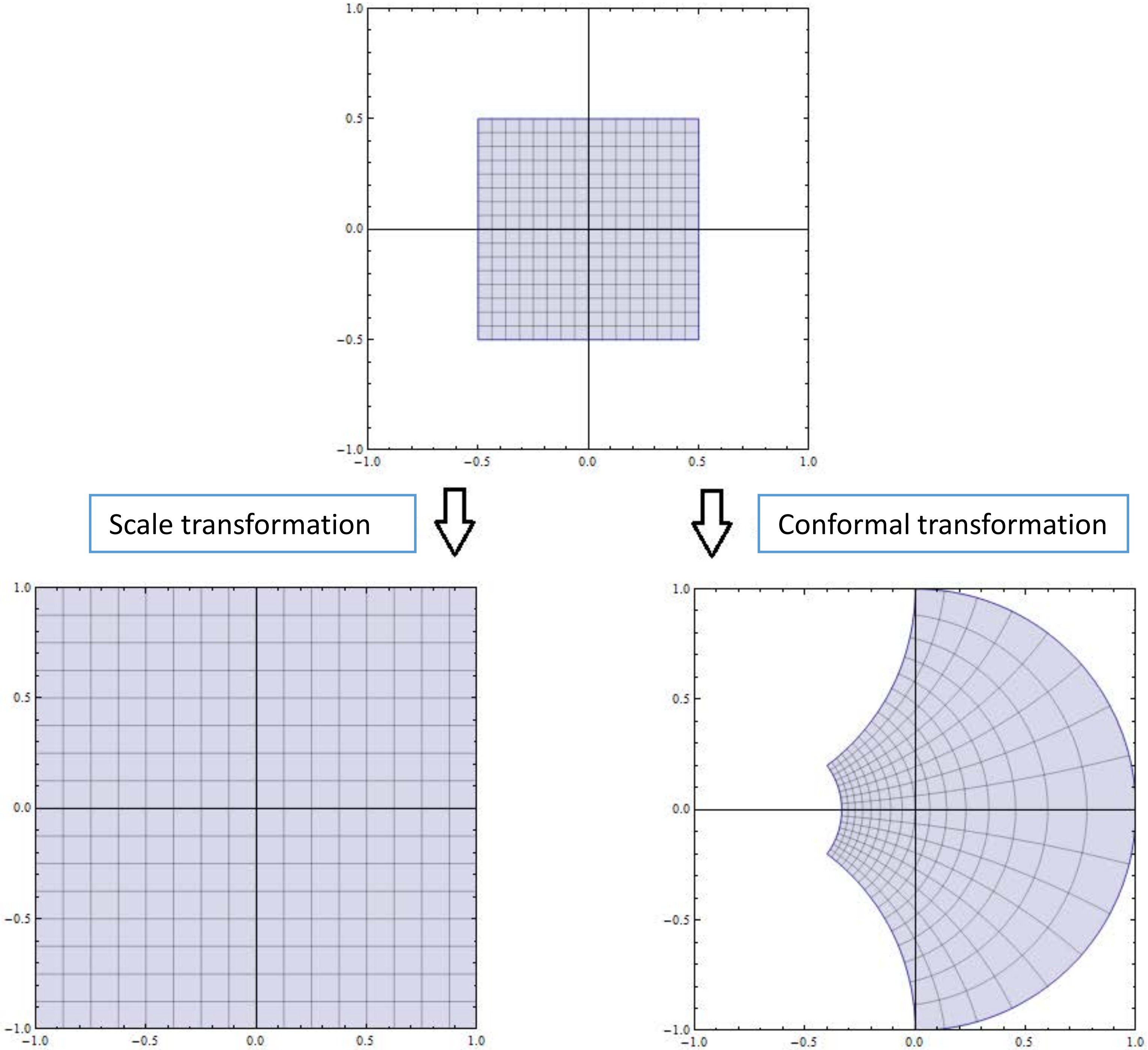
In 1908 Bateman and Cunningham discovered that the symmetry group of the Maxwell equations is, in fact, wider P ⇒ SO(4,2). For infinitesimal transformations

$$x^\mu \to x'^\mu = x^\mu + x_0^\mu + \Lambda^\mu{}_\nu x^\nu + \lambda x^\mu + 2(a\cdot x)x^\mu - x^2 a^\mu$$

scale transformation

special conformal transformation

scale and conformal transformations



Change of the metric

The Jacobian of the conformal coordinate transformation is proportional to Lorentz transformation,

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \Lambda^{\mu}{}_{\nu} b(x) \ ,$$

meaning that the metric transforms as:

$$g'_{\mu\nu}(x') = b^2(x)\eta_{\mu\nu} \ .$$

For instance, for dilatations $b(x) = \lambda$, and for special conformal transformations

$$b^2(x) = 1 + 2(a \cdot x) + a^2 x^2$$

Conformal field theories

CFT - quantum theories - are invariant with respect to conformal transformations, and the change of the (primary) fields with some internal spin s,

$$\phi(x) \to \tilde{\phi}(x') = b(x)^{-\Delta} R[\Lambda^{\mu}{}_{\nu}]\phi(x)$$

Here $R[\Lambda^{\mu}{}_{\nu}]$ is a representation matrix acting on the indices of $\phi(x)$ with spin s, and Δ is the scaling dimension of the field.

Important: scale invariance (part of the conformal group) forbids the presence of any dimensionful parameters in the theory.

Applications of CFTs

- Conformal symmetry restricts the form of different correlation functions.

- CFTs are an indispensable tool to address the behaviour of many systems in the vicinity of the critical points associated with phase transitions.

- CFTs describe the limiting behaviour of different quantum field theories deeply in the ultraviolet (UV) and/or infrared (IR) domains of energy.

Could it be that CFTs are even more important and that the ultimate theory of Nature is conformal?

Why this question?

CFTs may be relevant for the solution of the most puzzling fine-tuning issues of fundamental particle physics

- CFTs are theories without infinities.
- The Lagrangian of the Standard Model is invariant under the full conformal group (at the classical level) if the mass of the Higgs boson is put to zero. Perhaps, the observed smallness of the Fermi scale in comparison with the Planck scale is a consequence of this?
- The energy of the ground state in CFTs is equal to zero. Perhaps, this is relevant for the explanation of the amazing smallness of the cosmological constant?

Could it be that CFTs are even more important and that the ultimate theory of Nature is conformal?

Challenges:

Conformal invariance forbids the presence of any inherent dimensionful parameters in the action of a CFT. Because of that, CFTs have neither fundamental scales nor a well-defined notion of particle states. On the other hand, Nature has both.

Scale of quantum gravity, related to Newtons constant,
$G_N = 6.7 \times 10^{-39}$ GeV^{-2} , $M_P = 2.435 \times 10^{18}$ GeV

Fermi scale, associated with electroweak interactions,
$G_F = 1.17 \times 10^{-5}$ GeV^{-2}, $M_W = 80.38$ GeV

The cosmological constant, or vacuum energy, or Dark Energy,
$$\epsilon_{vac} = \left(2.24 \times 10^{-3}\mathrm{eV}\right)^4$$

Could it be that CFTs are even more important and that the ultimate theory of Nature is conformal?

Challenges:

Conformal symmetry is defined in the flat space-time. The ultimate theory must include gravity and arbitrary metric. A natural generalisation of conformal symmetry to curved space-time is the Weyl symmetry,

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}$$

However, this symmetry is anomalous if quantum corrections are taken into account! (Duff et al)

Hidden global symmetries

Hidden (or spontaneously broken symmetries) are well known in physics.

Lagrangian is invariant, but the ground state is not.

Example: U(1) global symmetry, theory of a single complex scalar field.

Different spectra:

- Symmetric phase - charged under U(1) scalar particle, 2 degrees of freedom.

- Broken phase: massive neutral particle and massless scalar particle, the Goldstone boson, 2 degrees of freedom.

Hidden conformal invariance

Classically scale-invariant Lagrangian

$$\mathscr{L} = \frac{1}{2}(\partial_\mu h)^2 + \frac{1}{2}(\partial_\mu \chi)^2 - V(h,\chi)$$

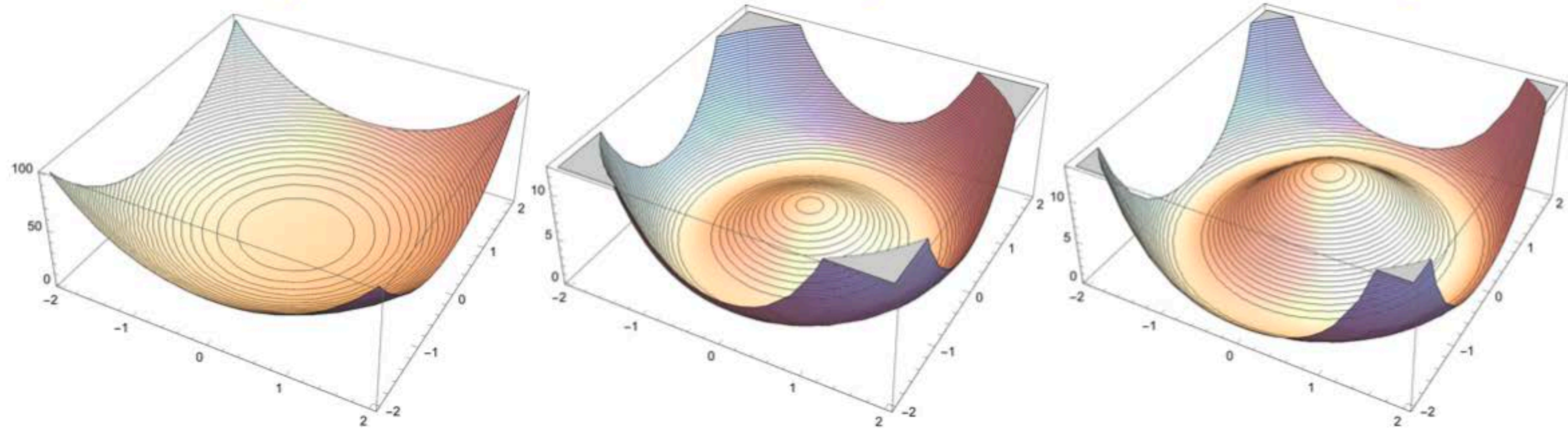
Potential:

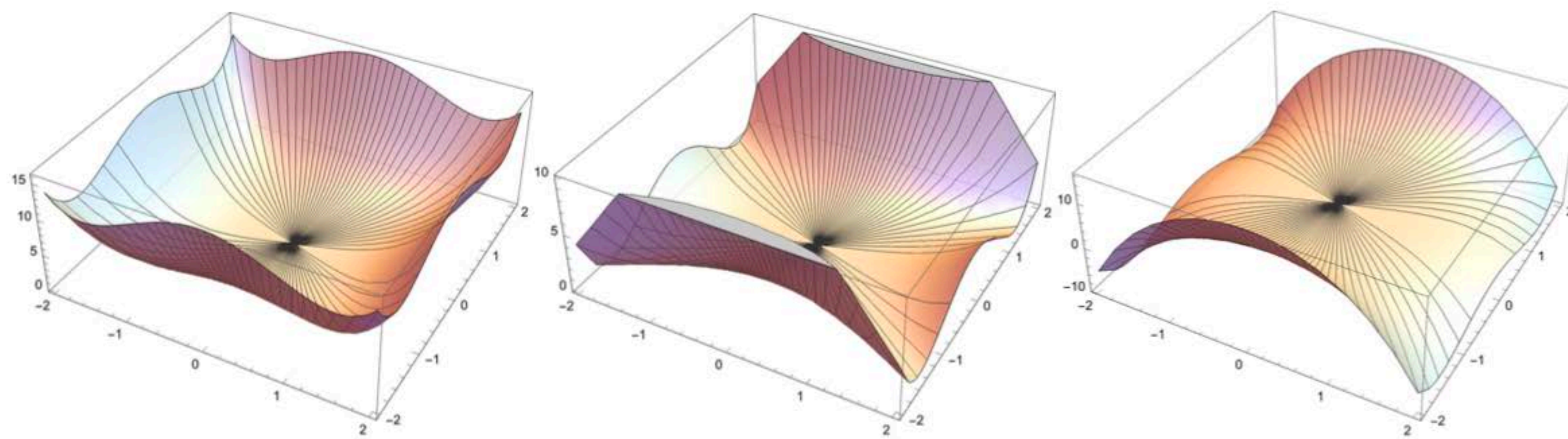
$$V(h,\chi) = \frac{\lambda}{4}\left(h^2 - \frac{\alpha}{\lambda}\chi^2\right)^2 + \beta\chi^4,$$

- $\beta > 0$: vacuum is unique, symmetry is exact, no particle excitations
- $\beta < 0$: vacuum is unstable
- $\beta = 0$: vacuum is degenerate, flat direction: $h^2 = \frac{\alpha}{\lambda}\chi^2$. Conformal symmetry is spontaneously broken, the spectrum consists of one massive particle - the Higgs boson, and one massless particle - the dilaton

Internal symmetry, $m^2(\phi_1^2+\phi_2^2)+\lambda(\phi_1^2+\phi_2^2)^2$

$m^2>0$ $\qquad$ $m^2<0$ $\qquad$ $m^2<0$





$\beta>0$ $\qquad$ $\beta=0$ $\qquad$ $\beta<0$

Scale symmetry, $\frac{\lambda}{4}\left(h^2-\frac{\alpha}{\lambda}\chi^2\right)^2+\beta\chi^4$

Conformally-invariant Standard Model

Lowest order effective action:

$$\mathscr{L} = \mathscr{L}_{\mathrm{SM}|\mathrm{m_H}\to 0} + \frac{1}{2}(\partial_\mu \chi)^2 - V(\varphi, \chi)$$

with potential

$$V(\varphi, \chi) = \lambda \left(\varphi^\dagger \varphi - \frac{\alpha}{2\lambda}\chi^2 \right)^2 + \beta \chi^4$$

- Flat direction, needed for spontaneous breaking of conformal symmetry: $\beta = 0$
- Particle spectrum: the same as in the Standard Model + massless dilaton. Phenomenologically, we must require $\alpha \ll 1$, i.e. $\langle \chi \rangle \gg 100$ GeV, to decouple the dilaton.

Construction of quantum conformal action

The standard renormalisation procedure always introduces some scale: cut-off Λ, or Pauli-Villars masses, or μ in dimensional regularisation, serving to solve for the mismatch between the units for coupling constants in spaces of different dimensionalities. This scale breaks conformal symmetry.

The way to keep the conformal symmetry intact all orders of perturbation theory: use the field-dependent cutoff or normalisation point: $\Lambda \propto \chi$, $\mu \propto \chi$ (Englert, Truffin, Gastmans, ’76, Wetterich ’88, Zenhäusern, M.S ’08, Gretsch, Monin, ’13)

Construction of quantum conformal action

The resulting theory is not renormalisable, one needs to add an infinite number of counter-terms.

However:

- For $\alpha \ll 1$ all counter-terms are suppressed by the dimensionful parameter $\langle\chi\rangle$
- We get an effective field theory valid up to the energy scale fixed by $\langle\chi\rangle$
- Gravity is non-renormalisable anyway, and making $\langle\chi\rangle \simeq M_P \simeq 10^{19}$ GeV does not make the theory worse

Low energy dilaton effective action

To see how gravity can be incorporated, we will need to have the low energy dilaton effective action. To find it, integrate out all massive states of the electroweak theory. Generic structure of conformally invariant terms:

- no derivatives: one operator, χ^4
- two derivatives, one operator $\chi \Box \chi$ - kinetic term
- four derivatives: two operators, $(\Box log(\chi))^2$ and $(\Box \chi)^2/\chi^2$ (related to the Weyl anomaly)
- see M.S. and Anna Tokareva e-Print: 2201.09232 [hep-th] for arbitrary power of the box.

Conformal symmetry and gravity

The natural extension of the conformal group to arbitrary metric: make the coordinate transformations $x' = F(x)$, which leave the metric $g_{\mu\nu}$ invariant up to a conformal factor $\Omega\,(x')$

$$g_{\mu\nu}(x) = \Omega\,(x')\,g'_{\lambda\sigma}\,(x')\,\frac{\partial F^{\lambda}}{\partial x^{\mu}}\frac{\partial F^{\sigma}}{\partial x^{\nu}}$$

"Undo" the coordinate transformation with the use of diffeomorphism invariance of general relativity. Get the Weyl transformation of the metric,

$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}$ and consider arbitrary $\Omega(x)$.

Looks like the Weyl symmetry is a natural extension of the conformal symmetry to curved space-time!

Quantum Weyl anomaly

- Consider the conformally invariant free theory of some matter fields (e.g. scalar or massless fermion, or electromagnetic field, etc) in flat space-time, couple it to gravity in a Weyl-invariant way, and integrate out the matter fields getting an effective action for the metric.

- Result (Duff et al, '77): the renormalised action is not Weyl-invariant. Weyl symmetry is anomalous and thus cannot be a symmetry of Nature. Perhaps, this can be cured by a specific particle content of the theory (supergravity, Fradkin and Tseytlin '84, other suggestions: Boyle and Turok '21).

Easy way to understand Weyl anomaly

Construction of Weyl-invariant theories with dilaton

- Take an arbitrary Diff-invariant action constructed from the metric only - the pure gravity

- Replace the metric $g_{\mu\nu}$ by $g_{\mu\nu}\chi^2/M_P^2$. This theory is Weyl-invariant (under transformation $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$, $\chi \to \Omega^{-1}\chi$) by construction. Example: $M_P^2 R \to \chi^2 R + 6(\partial\chi)^2$

- It is automatically conformal invariant if the metric is taken to be flat. Example: $M_P^2 R \to \chi^2 R + 6(\partial\chi)^2 \to 6(\partial\chi)^2$

- All operators of low energy dilaton theory in flat space-time, except one, can be constructed in this way

- This mismatch tells that this particular operator cannot be written in a Weyl-invariant way, meaning that the Weyl symmetry cannot be the exact symmetry of Nature

The troublesome operator

Let us attempt to construct the dilaton action in this way, considering operators with 4 derivatives. There are 4 and only 4 pure gravity invariants with 4 derivatives:

R^2, E_4, $W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}$, and $\Box R$

only one of them (R^2) will give a non-trivial result:

- W^2 is the Weyl invariant,
- the Euler density E_4 and $\Box R$ are the surface terms.

But we need to have 2 operators, $(\Box log(\chi))^2$ and $(\Box\chi)^2/\chi^2$!

The conjecture that Nature is conformally invariant is killed by gravity?

The potential flaw in the logic

- The conformal symmetry of flat space-time is the global symmetry, with a finite number of parameters characterising the transformation

- The Weyl symmetry is the local gauge symmetry with an infinite number of parameters (function) characterising the transformation

The correct question to ask:

Is there a finite subgroup of the Weyl group, which generates anomaly-free transformations, and coincides with the conformal group of Minkowski space-time if the metric is flat?

Anomaly free transformations

The troublesome action in flat space-time: $\int d^4x \tau \Box^2 \tau, \quad \tau = \log(\phi/\mu)$

Weyl-invariant generalisation for $D \neq 4$

$$\lim_{D\to 4} \int d^Dx \sqrt{-g} \left[\tau\Delta_4\tau + 2\tau\left(-\frac{1}{6}\Box R + \frac{1}{4}E_4\right) + \frac{R^2}{36} + L_{anom} \right]$$

$\Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}(\nabla^\mu R)\nabla_\mu$ Riegert operator, found by Fradkin and Tseytlin

Anomaly term is singular when $D = 4$, $L_{anom} = \dfrac{E_4}{2(D-4)}$

Weyl transform of anomaly term

The infinitesimal Weyl transformation $\Omega = 1 + \omega,\ \omega \ll 1$ of the anomaly term is

$$\delta_\omega L_{anom} = \frac{1}{2} E_4 \omega$$

There is no way to construct a local operator in $D = 4$ with Weyl transformation properties of L_{anom}.

Can we find ω for which $E_\omega = \int d^4x \sqrt{-g} E_4 \omega$ is surface integral for arbitrary metric?

Condition on anomaly-free Weyl transformation

Variation of E_ω with respect to the metric, $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$ after several integration by parts is

$$\delta E_\omega = \int d^4x\sqrt{-g}\,h_{\mu\nu}\Sigma^{\mu\nu\alpha\beta}\nabla_\alpha\nabla_\beta\,\omega$$

with

$$\Sigma^{\mu\nu\alpha\beta} = 2R(g^{\alpha\mu}g^{\beta\sigma} - g^{\alpha\beta}g^{\mu\nu}) + 4R^{\mu\nu}g^{\alpha\beta} + 4g^{\mu\nu}R^{\alpha\beta} - 8g^{\mu\beta}R^{\alpha\nu} - 4R^{\mu\alpha\nu\beta}$$

Anomaly is absent if and only if $\nabla_\alpha\nabla_\beta\,\omega = 0$, but for arbitrary metric, this means that $\omega = const$. Note that for flat metric the solution corresponds to conformal transformation, $\omega = c(1 + 2a_\mu x^\mu)$, with c corresponding to dilatations, and a_μ to special conformal transformations.

Equation for ω

$$\nabla_\alpha \nabla_\beta \omega = 0 \quad \longrightarrow \quad [\nabla_\gamma \nabla_\beta] \nabla_\alpha \omega = R_{\beta\gamma\lambda\alpha} \nabla^\lambda \omega = 0$$

$$R_{\alpha\beta} \nabla^\beta \omega = 0 \quad \longrightarrow \quad \omega = const$$

Conclusion:
Only scale symmetry may be a global symmetry of Nature.

Scale symmetry and phenomenology

The lowest order dilaton-gravity effective action

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}\zeta\chi^2 R + \frac{1}{2}(\partial_\mu\chi)^2 - \frac{\lambda}{4}\chi^4\right],$$

Important points:

- ζ is arbitrary (Weyl invariance would require -1/6).
- The Planck scale is dynamically generated when scale is spontaneously broken.
- The dilaton has only derivative couplings to matter and thus do not lead to any long-ranged fifth force.
- $\lambda \lll 1$ is required for the cosmological constant to be small. Action in the Einstein frame: Einstein-Hilbert action with massless scalar field and the cosmological constant $\propto \lambda/\zeta^2 M_P^4$.

Four derivatives scale invariant effective action

$$S = \int d^4x\sqrt{-g}\,\Big[AR\,\Box\,\tau + CR(\partial_\mu\tau)^2) + BG^{\mu\nu}\partial_\mu\tau\partial_\nu\tau +$$

$$+K\tau E_4 + ER^2 + FW^2_{\mu\nu\lambda\rho} + G((\partial_\mu\tau)^2)^2 +$$

$$+H(\Box\tau)^2 + J(\Box\tau + (\partial_\mu\tau)^2)^2\Big]$$

Here A, B, C, E, F, G, H, K and J are arbitrary dimensionless constants, $G_{\mu\nu}$ is the Einstein tensor, and $\tau = \log\chi$. Third line: operators admitted by conformal invariance in flat space-time.

“Complete” scale-invariant theory

Particles of the SM

+

graviton

+

dilaton

+

3 Majorana leptons

Scale-invariant Lagrangian

$$\mathscr{L}_{\nu\mathrm{MSM}} = \mathscr{L}_{\mathrm{SM[M\to 0]}} + \mathscr{L}_G + \frac{1}{2}(\partial_\mu \chi)^2 - V(\varphi, \chi) +$$

$$\left(\bar{N}_I i \gamma^\mu \partial_\mu N_I - h_{\alpha I} \bar{L}_\alpha N_I \tilde{\varphi} - f_I \bar{N}_I^{\,c} N_I \chi + \mathrm{h.c.} \right)$$

Gravity part: (φ - the Higgs field, $\varphi^\dagger \phi = h^2$)

$$\mathscr{L}_G = -\left(\xi_\chi \chi^2 + 2\xi_h \varphi^\dagger \varphi \right) \frac{R}{2}$$

Roles of different particles

Dilaton:

- determine the Planck mass
- give mass to the Higgs boson
- give masses to 3 Majorana leptons

Higgs Boson:

- give masses to fermions and vector bosons of the SM
- provide inflation

Majorana leptons:

- give masses to neutrinos
- provide dark matter candidate
- lead to baryon asymmetry of the Universe

Conclusions

- The global scale symmetry is the only anomaly-free extension of the conformal symmetry of flat space-time to curved space-time. It can be kept at a quantum level even if dynamical gravity is present.

- If spontaneously broken, the scale symmetry is consistent with all available experiments and observations and thus may play a role as the global symmetry of Nature.

Challenges

- What is the reason for the existence of the flat direction along the dilaton field, required for the very possibility of spontaneous breaking of scale symmetry? This is related closely to the cosmological constant problem.

- Yet another difficult task is building the connection between the low energy effective theory with spontaneously broken scale symmetry and its high energy limit hopefully leading to UV complete CFT with restored scale invariance.