Anomaly-induced effective action and Starobinsky model of inflation

Ilya L. Shapiro

Universidade Federal de Juiz de Fora, MG, Brazil

University of Geneva, Switzerland

Partially supported by: CNPq, FAPEMIG, ICTP

MPI-Heidelberg - January 12, 2015

Contents

- **Some examples of** $4d$ **conformal theories.**
- **Conformal anomaly and its ambiguities.**
- **Anomaly induced effective action.**
- **Light massive fields case.**
- **Applications:**
- **Vacuum states near black holes.**
- **Starobinsky model: unstable and stable versions.**

Examples of *4d* conformal theories

- **General scalar action with ξ term**

$$S_{scal} = \int d^4x\sqrt{-g}\,\left\{\frac{1}{2}\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\,\xi R\phi^2 - \frac{f}{4!}\,\phi^4\right\}$$

is invariant under global but not local conformal transformation.

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu}e^{2\lambda}\,, \qquad \phi \to \phi' = \phi\, e^{-\lambda}\,, \qquad \lambda = \textbf{const}\,.$$

Only in the case $\qquad \xi = \dfrac{1}{6}$

one meets local conformal symmetry

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu}\, e^{2\sigma}, \qquad \phi \to \phi' = \phi\, e^{-\sigma},$$

$$\sigma = \sigma(x)\,.$$

- **General metric-dilaton theory** *Shapiro & Takata, PLB-1994*

$$S = \int d^4x\sqrt{-g}\,\{A(\phi)\,(\nabla\phi)^2 + B(\phi)R + C(\phi)\}\,.$$

Consider conformal transformation of the metric plus scalar reparametrization

$$g'_{\mu\nu} = g_{\mu\nu}e^{2\sigma(\phi)}, \qquad\qquad \Phi = \Phi(\phi)$$

in O'Hanlon (Jackiw-Teitelboim - like) action

$$S = \int d^4x\sqrt{-g'}\,\{R'\Phi + V(\Phi)\}$$

Simple calculation gives $A(\phi) = 6e^{2\sigma(\phi)}[\Phi\sigma_1 + \Phi_1]\sigma_1$ **and**

$$B(\phi) = \Phi(\phi)e^{2\sigma(\phi)}\,, \quad C(\phi) = \left[\frac{B(\phi)}{\Phi(\phi)}\right]^2 V(\Phi(\phi))\,, \quad \textbf{where} \quad B_1 = \frac{dB}{d\phi}\,.$$

The conformal symmetry corresponds to the GR instead of JT.

$$\Phi = const \quad\Longrightarrow\quad A = \frac{3B_1^2}{2B}\,, \quad C = \lambda B^2\,.$$

The well-known particular case is

$$S = \int d^4x\sqrt{-g}\,\left\{-\frac{1}{2}\,\phi\,\Delta_2\,\phi - \frac{f}{4\,!}\,\phi^4\right\}$$

where

$$\Delta_2 = \Box + \frac{1}{6}\,R\,.$$

It is equivalent to Einstein-Hilbert action with a wrong sign

$$S_{EH} = +\,\frac{1}{16\pi G}\,\int d^4x\sqrt{-g}\,\{R + 2\Lambda\}\,.$$

The change of sign is perfectly possible, e.g., in the theory with torsion.

J. Helayel-Neto, A. Penna-Firme, I.Sh., PLB; gr-qc/9907081.

Massless spinor and vector fields

$$S_{1/2} = \frac{i}{2}\int d^4x\sqrt{-g}\,\{\bar{\psi}\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\,\gamma^\mu\psi\}$$

and

$$S_1 = -\frac{1}{4}\int d^4x\sqrt{-g}\,F_{\mu\nu}F^{\mu\nu}\,.$$

The transformation rules are

$$\psi \to \psi' = \psi\, e^{-3\sigma/2}\,, \quad \bar{\psi} \to \bar{\psi}' = \bar{\psi}\, e^{-3\sigma/2}\,, \quad A_\mu \to A'_\mu = A_\mu\,,$$

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu}\, e^{2\sigma} \qquad \sigma = \sigma(x)\,.$$

Note: the difference between conformal weight and dimension for the vector field is due to

$$A_\mu = A_b\, e^b_\mu\,, \quad e^b_\mu\, e^a_\nu\, \eta_{ab} = g_{\mu\nu}.$$

Direct relation between local & global conformal symmetries.

- **The conformal (Weyl) gravity in the dimension $n = 4$ includes only metric field**

$$S_W = \int d^4x \sqrt{-g}\, C^2 \,,$$

It can be easily generalized to an arbitrary dimension

$$C^2(n) = R^2_{\mu\nu\alpha\beta} - \frac{4}{n-2} R^2_{\mu\nu} + \frac{1}{(n-1)(n-2)} R^2 \,.$$

- **Fourth derivative scalars of the first kind**

$$S_4 = \int d^4x \sqrt{-g}\; \varphi \, \Delta_4 \, \varphi \,,$$

where $\Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3} R\Box + \frac{1}{3} R_{;\mu} \nabla^\mu \,.$

The transformation law is $\varphi \to \varphi' \,.$

S.M. Paneitz, MIT preprint - 1983; SIGMA - 2008
R.J. Riegert; E.S. Fradkin & A.A. Tseytlin, PLB - 1984.

- **Finally, there is a third derivative spinor field**

$$S_3 = \frac{i}{2}\int d^4x\sqrt{-g}\,\left\{\,\bar{\psi}\gamma^\mu\mathcal{D}_\mu\psi - \mathcal{D}_\mu\bar{\psi}\,\gamma^\mu\psi\,\right\}\,,$$

where

$$\mathcal{D}_\mu = \nabla_\mu\Box + R_{\mu\nu}\nabla^\nu - \frac{5}{12}R\nabla_\mu - \frac{1}{12}(\nabla_\mu R)\,.$$

The transformation law for the spinor ψ is

$$\psi \to \psi' = \psi\, e^{-\sigma/2}\,, \qquad \bar{\psi} \to \bar{\psi}' = \bar{\psi}\, e^{-\sigma/2}\,.$$

E.S. Fradkin & A.A. Tseytlin, Phys.Repts.-1985;
G. de Berredo Peixoto & I.Sh. PLB-2001.

Is it possible to construct more examples of conformal fields?
Vectors? Scalars? Spinors? Spin $3/2$?

Perhaps yes.

General Review:
V. Faraoni, E. Gunzig and P. Nardone, Fund. Cosmic Phys. **20** *(1999); [gr-qc/9811047].*

Open Problems (personal list).

- **Generalization of the conformal actions to higher dimensions (partially done, in simplest cases).**
Search and construction of the new conformal theories.

- **Conformal properties of topological terms, e.g., Gauss-Bonnet invariant in** n=4,

$$\int d^4x\sqrt{-g}E\,, \qquad E = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4\,R_{\alpha\beta}R^{\alpha\beta} + R^2\,.$$

- **Applications of conformal duality to cosmology and black-hole physics**

J. Bekenstein, Annals Phys. 82 (1974) 535.

I.Sh., Class. Quant. Grav. 14 (1997) 391;
A. de Barros and I.Sh., PLB 412 (1997) 242;
A. Barros, N. Pinto-Neto and I.Sh., Class. Q. Grav. 16 (1999) 1773.

Quantum (Semiclassical) Theory

Introduction: *Birrell & Davies (1980); Buchbinder, Odintsov & I.Sh. (1992).*

The most remarkable thing at the quantum level is that the classical conformal invariance is broken (trace anomaly).

Recent reviews: *I.Sh. et al. - gr-qc/0412113, hep-th/0610168 (both very technical), gr-qc/0801.0216.*

The first step is to consistently formulate the action on classical curved background.

In a conformal theory at 1-loop level it is sufficient to consider

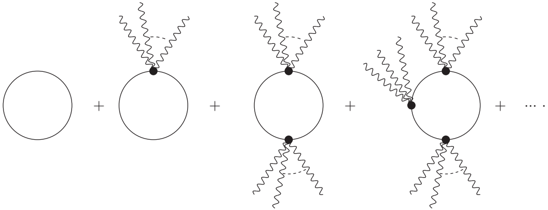
$$S_{conf.\ vac} = \int d^4x \sqrt{-g}\,\left\{a_1 C^2 + a_2 E + a_3 \Box R\right\}\ .$$

QFT in curved space can be renormalizable if we define

$$S_t = S_{min} + S_{non.min} + S_{vac} \,.$$

Renormalization involves fields and parameters like couplings and masses, ξ and vacuum action parameters.

Relevant diagrams for the vacuum sector



All possible covariant counterterms have the same structure as

$$S_{vac} \,=\, S_{EH} \,+\, S_{HD}\,, \qquad S_{EH} \,=\, -\,\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R + 2\Lambda)\,,$$

$$S_{HD} \,=\, \int d^4x\sqrt{-g}\,\{a_1C^2 + a_2E + a_3\Box R + a_4R^2\}\,.$$

Conformal anomaly

k_Φ **is the conformal weight of the field** Φ.
The Noether identity for the local conformal symmetry

$$\left[-2\, g_{\mu\nu}\,\frac{\delta}{\delta g_{\mu\nu}} + k_\Phi\,\Phi\,\frac{\delta}{\delta\Phi}\right]\, S(g_{\mu\nu},\,\Phi) \,=\, 0$$

produces on shell $$-\,\frac{2}{\sqrt{g}}\,g_{\mu\nu}\,\frac{\delta S_{vac}(g_{\mu\nu})}{\delta g_{\mu\nu}} \,=\, T_{(vac)}{}_{\mu}^{\;\mu} \,=\, T_\mu^\mu \,=\, 0\,.$$

At quantum level $S_{vac}(g_{\mu\nu})$ **is replaced by the EA** $\Gamma_{vac}(g_{\mu\nu})$.

For free fields only 1-loop order is relevant,

$$\Gamma_{div} \,=\, -\,\frac{1}{\varepsilon}\int d^4x\sqrt{g}\,\{\beta_1 C^2 + \beta_2 E + \beta_3 \Box R\}\;.$$

For the global conf. symmetry the renorm. group tells us

$$\langle T_\mu^\mu\rangle \,=\, \{\beta_1 C^2 + \beta_2 E + a'\Box R\}\;,$$

where $a' = \beta_3$. **In the local case** a' is **ambiguous.**

The simplest way to derive the conformal anomaly is using dimensional regularization (Duff, 1977).

The expression for divergences

$$\bar{\Gamma}_{div} = \frac{1}{\varepsilon} \int d^4x \sqrt{g} \left\{ \beta_1 C^2 + \beta_2 E + \beta_3 \Box R \right\} .$$

where

$$\begin{pmatrix} \beta_1 \\ -\beta_2 \\ \beta_3 \end{pmatrix} = \frac{1}{360(4\pi)^2} \begin{pmatrix} 3N_0 + 18N_{1/2} + 36N_1 \\ N_0 + 11N_{1/2} + 62N_1 \\ 2N_0 + 12N_{1/2} - 36N_1 \end{pmatrix}$$

The renormalized one-loop effective action has the form

$$\Gamma_R = S + \bar{\Gamma} + \Delta S \,,$$

where $\bar{\Gamma} = \bar{\Gamma}_{div} + \bar{\Gamma}_{fin}$ is the naive quantum correction to the classical action and ΔS is a counterterm.

ΔS **is an infinite local counterterm which is called to cancel the divergence. It is the only source of noninvariance**

The anomalous trace is

$$T \;=\; \langle T^\mu_\mu\rangle \;=\; -\,\frac{2}{\sqrt{-g}}\,g_{\mu\nu}\;\frac{\delta\,\Gamma_R}{\delta\,g_{\mu\nu}}\Big|_{n=4} \;=\; -\,\frac{2}{\sqrt{-g}}\,g_{\mu\nu}\;\frac{\delta\,\Delta S}{\delta\,g_{\mu\nu}}\Big|_{n=4}\,.$$

Conformal parametrization of the metric:

$$g_{\mu\nu} = \bar{g}_{\mu\nu}\cdot e^{2\sigma}\,, \qquad \sigma = \sigma(x)$$

where $\bar{g}_{\mu\nu}$ is the fiducial metric with fixed determinant.

There is a useful relation

$$\frac{2}{\sqrt{-g}}\,g_{\mu\nu}\,\frac{\delta\,A[g_{\mu\nu}]}{\delta\,g_{\mu\nu}} = \frac{1}{\sqrt{-\bar{g}}}\;\frac{\delta\,A[\bar{g}_{\mu\nu}\,e^{2\sigma}]}{\delta\sigma}\Big|_{\bar{g}_{\mu\nu}\to g_{\mu\nu},\,\sigma\to 0,\,n\to 4} \qquad (*)$$

$$\int d^nx\,\sqrt{-g}\,C^2(n) \;=\; \int d^nx\,\sqrt{-\bar{g}}\,e^{(n-4)\sigma}\,\bar{C}^2(n)\,.$$

Then $$\frac{\delta}{\delta\sigma}\int\frac{d^4x\sqrt{-\bar{g}}}{n-4}\,e^{(n-4)\sigma}\,\bar{C}^2(n)\Big|_{n\to 4} \;=\; \sqrt{-g}\,C^2\,.$$

The derivatives of $\sigma(x)$ in other terms are irrelevant.

In the simplest case $\sigma = \lambda = const$, we immediately arrive at the expression for T with $a' = \beta_3$.

For global conformal transform this procedure always works,

$$\langle T^\mu_\mu \rangle = \frac{1}{(4\pi)^2} \left(\omega C^2 + bE + c\Box R \right) .$$

However the local case $\sigma(x)$ it is more complicated, e.g.,

$$\frac{\delta}{\delta g_{\mu\nu}} \int \sqrt{-g}\Box R \equiv 0 .$$

We have a conflict between global and local conf. anomalies.

Or a conflict between formulas and intuitive expectations.
M.J. Duff, Class. Quantum. Grav. (1994)

Problem resolved:
M.Asorey, E. Gorbar & I.Sh., CQG 21 (2003).

Anomaly-induced Effective Action (EA) of vacuum

One can use $\langle T^\mu_\mu \rangle$ to obtain equation for the finite 1-loop EA

$$\frac{2}{\sqrt{-g}}\, g_{\mu\nu} \frac{\delta\, \bar{\Gamma}_{ind}}{\delta g_{\mu\nu}} = \langle T^\mu_\mu \rangle = \frac{1}{(4\pi)^2} \,\big(\omega C^2 + bE + c\Box R \big)\ .$$

The solution is straightforward

Riegert; Fradkin & Tseytlin, PLB-1984.

It can be generalized for the theory with more background fields, e.g., with vector, torsion or scalar fields.

Buchbinder, Odintsov & I.Sh. Phys.Lett. B (1985).
Helayel-Neto, Penna-Firme & I.Sh. Phys.Lett. B (1998);
I.Sh., J. Solà, Phys.Lett. B (2002);
M. Giannotti, E. Mottola, Phys. Rev. D (2009).

The simplest possibility is to parameterize metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} \cdot e^{2\sigma}\,, \qquad \sigma = \sigma(x)\,.$$

The solution for the effective action is

$$\bar{\Gamma}_{ind} \;=\; S_c[\bar{g}_{\mu\nu}] \,+\, \frac{1}{(4\pi)^2}\,\int d^4x\,\sqrt{-\bar{g}}\,\{\omega\sigma\bar{C}^2 \qquad (1)$$

$$+\, b\sigma(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R}) + 2b\sigma\bar{\Delta}_4\sigma \,-\, \frac{1}{12}\,(c + \frac{2}{3}b)[\bar{R} - 6(\bar{\nabla}\sigma)^2 - (\bar{\Box}\sigma)]^2)\}\,,$$

where $S_c[\bar{g}_{\mu\nu}] = S_c[g_{\mu\nu}]$ is an unknown conformal functional, which serves as an integration constant in eq. for Γ_{ind}.

The solution (1) has serious merits:
1) Being simple, 2) Being exact in case $S_c[\bar{g}_{\mu\nu}]$ is irrelevant.
Example: FRW metrics.

An important disadvantage is that it is not covariant or, in other words, it is not expressed in terms of original metric $g_{\mu\nu}$.

Now we obtain the non-local covariant solution and after represent it in the local form using auxiliary fields.

First one has to establish the relations

$$\sqrt{-g}C^2 = \sqrt{-\bar{g}}\bar{C}^2\,, \qquad \sqrt{-\bar{g}}\,\bar{\Delta}_4 = \sqrt{-g}\,\Delta_4\,,$$

$$\sqrt{-g}(E - \frac{2}{3}\Box R) = \sqrt{-\bar{g}}(\bar{E} - \frac{2}{3}\bar{\Box}\bar{R} + 4\bar{\Delta}_4\sigma)$$

and also introduce the Green function

$$\Delta_4\, G(x,y) = \delta(x,y)\,.$$

Using these formulas we find, for a functional $A(g_{\mu\nu}) = A(\bar{g}_{\mu\nu})$,

$$\frac{\delta}{\delta\sigma}\int_x A\;(E - \frac{2}{3}\Box R)\Big| \;=\; 4\sqrt{-g}\Delta_4\, A\,.$$

where $\int_x = \int d^4x\sqrt{-g(x)}\,, \qquad \Big| = \Big|_{\bar{g}_{\mu\nu}\to g_{\mu\nu}}$

As a consequence, we obtain

$$\frac{\delta}{\delta\sigma(y)} \iint_{x\,y} \frac{1}{4}\, C^2(x)\, G(x,y)\, \Big(E - \frac{2}{3}\Box R\Big)_y \Big|$$

$$= \int d^4x \sqrt{-\bar{g}(x)}\, \bar{\Delta}_4(x)\, \bar{G}(x,y)\, \bar{C}^2(x)\Big| \;=\; \sqrt{-g}\, C^2(y)\,.$$

Hence, the part of Γ_{ind} which is responsible for $T_\omega = -\omega C^2$, is

$$\Gamma_\omega = \frac{\omega}{4} \iint_{x\,y} C^2(x)\, G(x,y)\, (E - \frac{2}{3}\Box R)_y\,.$$

Similarly one can check that the variation $T_b = b\,(E - \frac{2}{3}\Box R)$ is produced by the term

$$\Gamma_b = \frac{b}{8} \iint_{x\,y} (E - \frac{2}{3}\Box R)_x\, G(x,y)\, (E - \frac{2}{3}\Box R)_y\,.$$

Finally, we can use simple relation

$$g_{\mu\nu}\,\frac{\delta}{\delta g_{\mu\nu}}\,\int_x R^2(x) \,=\, -6\,\sqrt{-g}\,\Box R\,.$$

to establish the remaining local constituent of Γ_{ind}

$$\Gamma_c = -\frac{3c+2b}{36(4\pi)^2}\,\int_x R^2(x)\,.$$

The general covariant solution for Γ_{ind} is the sum,

$$\Gamma_{ind} = S_c[g_{\mu\nu}] \,-\, \frac{3c+2b}{36(4\pi)^2}\,\int_x R^2(x)$$
$$+\,\frac{\omega}{4}\,\int_x\int_y C^2(x)\,G(x,y)\,(E-\frac{2}{3}\Box R)_y$$
$$+\,\frac{b}{8}\,\int_x\int_y (E-\frac{2}{3}\Box R)_x\,G(x,y)\,(E-\frac{2}{3}\Box R)_y\,.$$

One can rewrite this expression using auxiliary scalars.

The nonlocal terms can be rewritten in a symmetric form

$$\Big(E - \frac{2}{3}\Box R\Big)_x\, G(x,y)\, \Big[\frac{\omega}{4}C^2 - \frac{b}{8}(E - \frac{2}{3}\Box R)\Big]_y$$

$$= \frac{b}{8}\iint\limits_{x\,y} \Big(E - \frac{2}{3}\Box R - \frac{\omega}{b}C^2\Big)_x G(x,y)\Big(E - \frac{2}{3}\Box R - \frac{\omega}{b}C^2\Big)_y$$

$$- \frac{\omega^2}{8b}\iint\limits_{x\,y} C_x^2\, G(x,y)\, C_y^2\,.$$

These form is appropriate for rewriting it via auxiliary fields.
Then we arrive at the local covariant expression for EA

$$\Gamma_{ind} = S_c[g_{\mu\nu}] - \frac{3c+2b}{36(4\pi)^2}\int\limits_x R^2(x) + \int\limits_x \Big\{\frac{1}{2}\,\varphi\Delta_4\varphi - \frac{1}{2}\,\psi\Delta_4\psi$$

$$+ \frac{\omega}{8\pi\sqrt{-b}}\,\psi\, C^2 + \varphi\,\Big[\,\frac{\sqrt{-b}}{8\pi}\,(E - \frac{2}{3}\,\Box R) - \frac{\omega}{8\pi\sqrt{-b}}\,C^2\Big]\Big\}\,.$$

The above form of EA is the best one for Γ_{ind}.
I.Sh. and A.Jacksenaev, Phys. Lett. B (1994)

Similar expression has been independently introduced by
P. Mazur & E. Mottola, 1997-1998.

Some remarks are in order.

1) Imposing boundary conditions on two auxiliary fields φ **and** ψ **is equivalent to defining boundary conditions for the Green functions** $G(x,y)$.

2) Introducing the new term $\int C_x^2 G(x,y) C_y^2$ **into the action may be viewed as redefinition of the conformal functional** $S_c[g_{\mu\nu}]$.

However, writing the non-conformal terms in the symmetric form, essentially modifies the four-point function. Using ψ **we restore the structure generated by anomaly.**

Applications of the anomaly-induced EA

- **Classification of vacuum states in the vicinity of a black hole**

Anomaly is, in part, responsible for the Hawking radiation
S.M. Christensen, S.A. Fulling, PRD (1977).

The anomaly-induced effective action of gravity enables one to perform a kind of systematic classification of the vacuum states for the quantum fields on the black hole background.

We can distinguish the different vacuum states by choosing different boundary conditions for the auxiliary fields φ and ψ.

R. Balbinot, A. Fabbri & I.Sh., PRL **83**; *NPB* **559** *(1999).*

Generalization for the Reissner-Nordstrom black hole,
P.R. Anderson, E. Mottola & R. Vaulin, PRD **76** *(2007).*

At the classical level, the black hole (BH) does not emit radiation, but such emission can take place if we take quantum effects into account.

After being discovered by Hawking (1975)**, the same result has been obtained from analytical estimates of $\langle T_{\mu\nu} \rangle$ for quantum matter fields in a fixed Schwarzschild BH geometry.**

S.M. Christensen & S.A. Fulling, PRD 15 (1977).

Detailed analytical and numerical study, based on the analysis of $\langle T_{\mu\nu} \rangle$ in the classical black hole background:

P. Candelas, PRD **21** *(1980);*
D.N. Page, PRD **25** *(1982);*
M.R. Brown, A.C. Ottewill and D.N. Page, PRD **33** *(1986);*
V.P. Frolov and A.I. Zelnikov, PRD **35** *(1987);*
P.R. Anderson, W.A. Hiscock and D.A. Samuel, PRD **51** *(1995). ...*

A fundamental property is the existence of three different vacuum quantum states.

i) **The Boulware $|B\rangle$ state reproduces the Minkowski vacuum $|M\rangle$ in the limit $r \to \infty$, where $\langle B|T_{\mu\nu}|B\rangle \sim r^{-6}$.**
On the horizon this quantity is divergent in a free falling frame.

ii) **For Unruh vacuum $|U\rangle$ the value $\langle U|T_{\mu\nu}|U\rangle$ is regular on the future event horizon but not on the past one. Asymptotically in the future $\langle U|T_{\mu\nu}|U\rangle$ has the form of a flux of radiation at the Hawking temperature $T_H = 1/8\pi M$.**
This vacuum state is the most appropriate to discuss evaporation of black holes formed by gravitational collapse of matter.

iii) **The Israel-Hartle-Hawking $|H\rangle$** state
$\langle H|T_{\mu\nu}|H\rangle$ **for $r \to \infty$ describes a thermal bath of radiation at T_H.**

The existence of three vacuum states reflects distinct positions of observers and the construction of different *in* **and** *out* **modes with respect to the corresponding coordinates.**

The main difference between classical and quantum theories is that, in the first case we know how to transform the relevant quantities when we change the coordinate system.

The natural question is how to perform a transition between different vacuum states $|H\rangle$, $|B\rangle$ **and** $|U\rangle$ **?**

The anomaly-induced effective action doesn't make any reference to a particular quantum state, but it includes the conformal invariant functional $S_c[g_{\mu\nu}]$ **– a source of uncertainty.**

Strategy: one has to fix the extended set of boundary conditions, including the ones for the auxiliary scalars φ and ψ.

The procedure for identifying the vacuum state is as follows:

1) Solving equations for φ and ψ.
The solutions always depend on the set of integration constants.

2) One has to find "appropriate" boundary conditions to identify $\langle V|T_{\mu\nu}|V\rangle$ for the given vacuum state $|V\rangle = (|B\rangle,\ |U\rangle,\ |H\rangle)$.

3) Use

$$\langle T_{\mu\nu}\rangle \quad \longrightarrow \quad \frac{2}{\sqrt{-g}}\frac{\delta\Gamma_{ind}}{\delta g_{\mu\nu}} = \langle S_{\mu\nu}\rangle$$

where of course $\langle S\rangle = \langle T\rangle$.

The general solution is $\phi(r,t) = d\cdot t + w(r)$**, where** $w(r)$ **satisfies the equation**

$$\frac{dw}{dr} = \frac{B}{3}r + \frac{2MB}{3} - \frac{A}{6} - \frac{\alpha}{72M} + \frac{1}{r-2M}\Big(\frac{4}{3}BM^2 + \frac{C}{2M} - AM - \frac{\alpha}{24}\Big) - \frac{C}{2Mr}$$

$$- \Big[\frac{\alpha M}{r^3} + \frac{24AM-\alpha}{144\,M^2}\Big]\,\frac{r^2\,\ln r}{r-2M} + \frac{(24AM-a)\big(r^3-8M^3\big)\,\ln(r-2M)}{3r(r-2M)48M^2}\,.$$

(d,A,B,C) **are constants which specify the homogeneous solution** $\Box^2\phi = 0$ **and hence the quantum state.**

For ψ **we have a similar solution, but with** (d',A',B',C')**.**

Due to the independence of φ **and** ψ**. the two sets are independent on each other.**

In case of a Boulware state $|B\rangle$ we request

$$|B\rangle \to |M\rangle \quad \text{when} \quad r \to \infty\,.$$

In the Minkowski vacuum we can safely set $\varphi = \psi = 0$.

This asymptotic conditions enables one to arrive at the asymptotic expressions

$$\langle B|S^\nu_\mu|B\rangle \,\to\, \frac{\alpha^2 - \beta^2}{2\,(24)^2\,(2M)^4\,\big(1 - 2M/r\big)^2} \times \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 \end{pmatrix}$$

for $r \to 2M$ and

$$\langle B|S^\nu_\mu|B\rangle \propto \mathcal{O}\big(r^{-6}\big) \qquad \text{for} \qquad r \to \infty\,.$$

This behavior fits perfectly will with the ones observed within other methods.

Unruh vacuum case

Choosing another values of the integration constants we meet the following asymptotic behavior near the horizon $r \to 2M$:

$$\langle U|S_a{}^b|U\rangle \sim \frac{\alpha^2-\beta^2}{2(48M^2)^2}\begin{pmatrix} 1/f & -1 \\ 1/f^2 & -1/f \end{pmatrix},$$

regular on the future horizon, $a, b = r, t$. The asymptotic form

$$r \to \infty \qquad \langle U|S_\mu{}^\nu|U\rangle \to \frac{\alpha^2-\beta^2}{2r^2(24M)^2}\begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

These results are in exact agreement with the standard ones on the Hawking radiation: *B.S. DeWitt, Phys. Rep.* **C19** *(1975) 297.*

once the luminosity L of the radiating BH is identified with

$$\frac{L}{4\pi} = \frac{(\alpha^2-\beta^2)}{2(24M)^2}.$$

A bit more complicated situation takes place for the Hartle-Hawking vacuum.

One should not only properly choose initial conditions but also fine-tune the coefficient l_1 to order to achieve correspondence with the results achieved by other methods.

From the general perspective this situation looks somehow natural, especially because the unknown conformal invariant functional S_c is relevant for the spherically symmetric metric.

In case of the Hartle-Hawking vacuum, when the thermodynamic aspects play more important role, one can not really expect that the conformal anomaly gives complete description of the situation.

Cosmological application: Starobinsky Model.

Starobinsky model based on quantum effects.

Fischetti, Hartle and Hu (1978);
Starobinsky, (1980-1983) ;
Mukhanov, Chibisov, (1982) ;
Anderson, Vilenkin, ... (1983-1986)
Hawking, Hertog and Real, (2001) .

Modified Starobinsky model

Fabris, Pelinson, Solà, I.Sh., ...

Cosmological Model based on the action

$$S_{total} = -\frac{M_P^2}{16\pi}\int d^4x\sqrt{-g}\,(R+2\Lambda) + S_{matter} + S_{vac} + \bar{\Gamma}_{ind}\,.$$

Equation of motion for $a(t),\quad dt = a\,(\eta)\,d\eta,\quad k=0$

$$\frac{\dddot{a}}{a} + \frac{3\dot{a}\ddot{a}}{a^2} + \frac{\ddot{a}^2}{a^2} - \left(5+\frac{4b}{c}\right)\frac{\ddot{a}\dot{a}^2}{a^3} - \frac{M_P^2}{8\pi c}\left(\frac{\ddot{a}}{a} + \frac{\ddot{a}^2}{a^2} - \frac{2\Lambda}{3}\right) = 0\,,$$

$k = 0, \pm 1$. **Particular solutions (Starobinsky, PLB-1980)**

$$a(t) = a_0 e^{Ht}\,, \qquad k = 0\,,$$

where Hubble parameter $H = \dot{a}/a$ **is**

$$H^2 = -\frac{M_P^2}{32\pi b}\left(1 \pm \sqrt{1 + \frac{64\pi b}{3}\,\frac{\Lambda}{M_P^2}}\,\right).$$

A. Pelinson, I.Sh., F. Takakura, NPB (2003).

For $0 < \Lambda \ll M_P^2$ there are two solutions:

$$H \approx \sqrt{\Lambda/3}\,; \qquad (IR)$$

$$H \approx \sqrt{-\frac{M_P^2}{16\pi b} - \frac{\Lambda}{3}} \approx \frac{M_P}{\sqrt{-16\pi b}}\,. \qquad (UV)$$

Perturbations of the conformal factor

$$\sigma(t) \to \sigma(t) + y(t).$$

The criterion for a stable (UV) inflation is

$$c > 0 \quad \Longleftrightarrow \quad N_1 < \frac{1}{3}N_{1/2} + \frac{1}{18}N_0\,,$$

in agreement with Starobinsky (1980).

The original Starobinsky model is based on the unstable case and involves special choice of initial data. This situation can be improved further by using the stable version and an appropriate transition scheme.

Simple test of the unstable version of Starobinsky Model.
A.Pelinson, I.Sh. et al., IRGA-NPB(PS)- 2003.,

Consider late Universe, $k = 0, \; H_0 = \sqrt{\Lambda/3}$.

Only photon is active, $N_0 = 0\,, \qquad N_{1/2} = 0\,, \qquad N_1 = 1\,.$

Graviton typical energy is $H_0 \approx 10^{-42}\,GeV$, $\Longrightarrow$ **all massive particles (even neutrino)** $m_\nu \geq 10^{-12}\,GeV$ **decouple from gravity.** $c < 0 \quad \Longrightarrow$ **today inflation is unstable.**

Stability for the small $H = H_0$ **case:** $H \to H_0 + const \cdot e^{\lambda t}$

$$\lambda^3 + 7H_0\lambda^2 + \left[\frac{(3c-b)4{H_0}^2}{c} - \frac{M_P^2}{8\pi c}\right]\lambda - \frac{32\pi b{H_0}^3 + M_P^2 H_0}{2\pi c} = 0\,.$$

The solutions are $\lambda_1 = -4H_0\,, \quad \lambda_{2/3} = -\frac{3}{2}\,H_0 \pm \frac{M_P}{\sqrt{8\pi|c|}}\,i\,.$

$\Lambda > 0$ **protects our world from quantum corrections!**

Transition. **Suppose at UV** $(H \gg M_F)$ **there is SUSY, e.g. MSSM,**

$$N_1 = 12\,, \qquad N_{1/2} = 32\,, \qquad N_0 = 104\,.$$

This provides stable inflation, because

$$\frac{1}{3}\,N_{1/2} + \frac{1}{18}\,N_0 > N_1 \qquad \Longrightarrow \qquad c > 0\,.$$

For realistic SUSY model inflation is independent on initial data.

Fine!

But why should inflation end? **Already for MSM**
$(\,N_{1,\,1/2,\,0} = 12, 24, 4)$, $c < 0$, **inflation is unstable.**

Natural interpretation:
I.Sh. Int.J.Mod.Ph.D. (2002); A. Pelinson et al NPB (2003).

All sparticles are heavy $\Rightarrow$ **decouple when** H **becomes smaller than their masses.**

Direct calculations confirmed that the transition $c > 0 \Longrightarrow c < 0$ **is smooth, indicating a possibility of a smooth graceful exit.**

• Using anomaly for deriving EA of massive fields.

Question: Why the energy scale H decreases during inflation?

In the exponential phase Hubble parameter $H(t) = const$.

Another unclear point: Using anomaly-induced EA for massive fields is not a correct approximation.

Maybe all difficulties can be solved if taking masses of the fields into account?

Consider a reliable Ansatz for the EA of massive fields.
J.Solà, I.Sh. PLB - 2002;
also A.Pelinson, I.Sh. & F.Takakura, Nucl.Ph. 648B (2003).

In part, it is based on
R.D.Peccei, J.Solà, C.Wetterich, Ph.Lett. B 195 (1987) 183
and S. Deser, Ann. Phys. **59** *(1970) 248.*

The idea is to construct the conformal formulation of the SM and use it to derive EA for massive fields.

Conformal formulation of massive theory

The conformally non-invariant terms:

$$m_s^2\,\varphi^2\,,\quad m_f\,\bar{\psi}\psi\,,\quad \textbf{and}\quad L_{EH}\;=\;-\,\frac{1}{16\pi G}\,(R+2\Lambda)\,.$$

Replacing dimensional parameters by the new scalar χ:

$$m_{s,f}\to\frac{m_{s,f}}{M}\,\chi\,,\quad M_P^2\to\frac{M_P^2}{M^2}\,\chi^2\,,\quad \Lambda\to\frac{\Lambda}{M^2}\,\chi^2\,.$$

M **is related to a scale of conformal symmetry breaking.**
Massive terms get replaced by Yukawa and $(scalar)^4$ **type interactions with** χ**. In the IR** $\chi\sim M$.

In the gravity sector

$$\mathcal{L}_{EH}^{*}=-\frac{M_P^2}{16\pi\,M^2}\,\left\{[R\chi^2+6\,(\partial\chi)^2]\,+\,\frac{2\Lambda\,\chi^4}{M^2}\right\}$$

in order to provide local conformal invariance.

The new theory is conformal invariant

$$\sigma = \sigma(x), \qquad \begin{cases} \chi \to \chi\, e^{-\sigma}\,, & g_{\mu\nu} \to g_{\mu\nu}\, e^{2\sigma} \\ \varphi \to \varphi\, e^{-\sigma}\,, & \psi \to \psi\, e^{-3/2\,\sigma} \end{cases}$$

The conformal symmetry comes together with a new scalar χ, absorbing conformal degree of freedom. Fixing $\chi \to M$ we come back to original formulation.
The conformal anomaly becomes

$$\langle T \rangle = -\Big\{ wC^2 + bE + c\Box R \; + \; \frac{f}{M^2}\,[R\chi^2 + 6(\partial\chi)^2] + \frac{g}{M^4}\,\chi^4 \Big\},$$

f **and** g **are** β**-functions for** $(16\pi G)^{-1}$ **and** $\rho_\Lambda = \Lambda/8\pi G$ **,**

$$f \;=\; \sum_i \frac{N_f}{3\,(4\pi)^2}\, m_f^2\,, \qquad \tilde{f} = \frac{16\pi f}{M_P^2}\,,$$

$$g \;=\; \frac{1}{2(4\pi)^2} \sum_s N_s\, m_s^4 - \frac{2}{(4\pi)^2} \sum_f N_f\, m_f^4\,,$$

N_f **and** N_s **are multiplicities of the fields.**

Anomaly-induced EA in terms of $g_{\mu\nu} = \bar{g}_{\mu\nu} \cdot e^{2\sigma}$ and $\chi = \bar{\chi} \cdot e^{-\sigma}$

$$\bar{\Gamma} \,=\, S_c[\bar{g}_{\mu\nu},\bar{\chi}] \,+\, \int d^4x\sqrt{-\bar{g}}\Big\{w\sigma\bar{C}^2 + b\sigma(\bar{E} - \frac{2}{3}\bar{\nabla}^2\bar{R}) + 2b\sigma\bar{\Delta}\sigma$$

$$+\, \frac{f}{M^4}\,\sigma[\bar{R}\bar{\chi}^2 + 6(\partial\bar{\chi})^2] \,+\, \frac{g}{M^4}\,\bar{\chi}^4\sigma\,\Big\} - \frac{3c+2b}{36}\int d^4x\sqrt{-g}R^2\,.$$

This may be seen as a local version of Renormalization Group.
In curved space-time RG corresponds to the scaling

$$g_{\mu\nu} \to g_{\mu\nu}\cdot e^{-2\tau} \quad\Longrightarrow\quad \Gamma[e^{-2\tau}g_{\alpha\beta},\Phi_i,P,\mu] = \Gamma[g_{\alpha\beta},\Phi_i(\tau),P(\tau),\mu]\,.$$

In the leading-log approximation we meet the RG improved classical action of vacuum

$$S_{vac}[g_{\alpha\beta},P(\tau),\mu]\,, \quad \textbf{where} \quad P(\tau) = P_0 + \beta_P\,\tau\,.$$

The equivalence in all terms which do not vanish for $\sigma = \tau$.

Cosmological implications

$$S_t = S_{matter} + S^*_{EH} + S_{vac} + \bar{\Gamma} \,.$$

The equation of motion for $\Lambda = 0,\ g = 0$

$$a^2\,\dddot{a} + 3\,a\,\dot{a}\,\ddot{a} - \left(5 + \frac{4b}{c}\right)\dot{a}^2\,\ddot{a} + a\,\ddot{a}^2 - \frac{M_P^2}{8\pi c}\left(a^2\ddot{a} + a\dot{a}^2\right)\,[1 - \tilde{f}\cdot\ln a] \;=\; 0\,,$$

Let us solve by $M_P^2 \to M_P^2\,[1 - \tilde{f}\cdot\ln a]$,

$$\dot{\sigma} = H = H_o\sqrt{1 - \tilde{f}\sigma(t)}\,, \qquad H_o = \frac{M_P}{\sqrt{-16b}}\,.$$

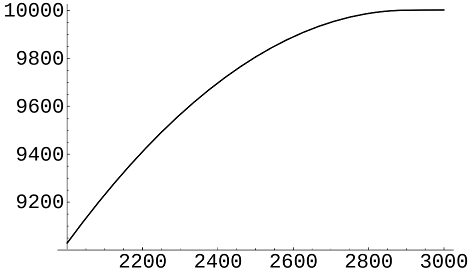
This leads to the simple solution

$$\sigma(t) \;=\; H_o\,t \;-\; \frac{H_0^2}{4}\,\tilde{f}\;t^2\,.$$

Remarkably, this formula fits with the numerical solution with a wonderful 10^{-6} **precision!**

$\tilde{f} > 0 \;\;\Rightarrow\;\;$ **we arrive at the tempered inflation!!**

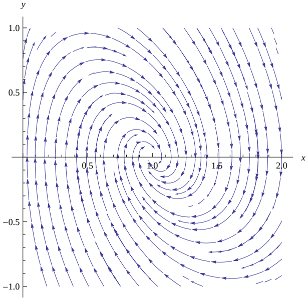
Anomaly-induced inflation slows down if taking masses of quantum fields into account.



$$\sigma(t) = \ln a(t) \approx H_0 t - \frac{H_0^2}{4} \tilde{f} \, t^2 , \qquad H_0 \propto M_P$$

The total amount of *e*-folds may be as large as 10^{32}, but only 65 last ones, where $H \propto M_*$ (SUSY breaking scale) are relevant.

Remember that in the unstable phase there are very different solutions, some of them violent (hyperinflation). How can we know that the transition from stable to unstable phase really happens? *A. Pelinson et al, NPB(PS) (2003).* **Phase portrait:**



$$\text{Starobinsky (1980)}: \; x = \left(\frac{H}{H_0}\right)^{\frac{3}{2}}, \quad y = \frac{\dot{H}}{2\sqrt{H_0^3 H}}, \quad dt = \frac{dx}{3H_0 x^{2/3} y}.$$

From the formal QFT viewpoint, there is no solution, because for the transition period, when

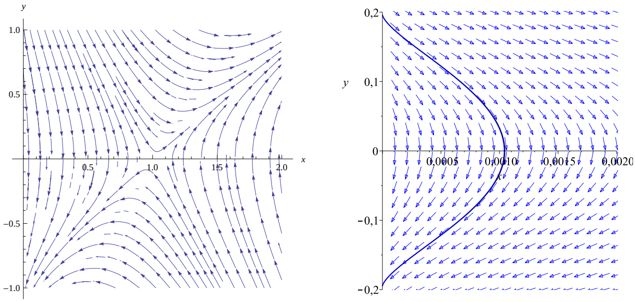
$$H \quad \sim \quad \text{masses of quantum matter fields}$$

we have no method, approach, idea or approximation to perform calculations, except for dS space, which is useless here.

The simplest, purely phenomenological approach is to take a final point of the stable tempered inflation epoch ... and use it as initial point for the unstable phase. Where we are going to end up in this way?

A. Pelinson, T. de Paula, I.Sh., A. Starobinsky, paper is preparation.

The qualitative output of this phenomenological approach is positive, in the sense that the final point of the stable inflation (related to SUSY breakdown) belongs to the "right" integration curve of the unstable inflation.



One can check that this curve really ends up at the classical radiation-dominated solution.

This result gives us a chance to have a consistent inflation based on QFT results.

Other features of the Modified Starobinsky Model.

In the last 65 e-folds the production of gravitational waves is restricted

$$H(t) \ll 10^{-5} M_P .$$

After being created, gravitational waves do not amplify.

A.A.Starobinski, Let.Astr.Journ. 9 (1983);

J. Fabris, A. Pelinson, I.Sh., Nucl.Phys. 597B (2001);
J. Fabris, A. Pelinson, I.Sh, F.Takakura, NPB(PS) 127 (2004);
J. Fabris, A. Pelinson, F. Salles, I.Sh, arXiv:1112.5202 (JCAP).

From the QFT viewpoint the main problem is that too small information is available about intermediate stage of inflation.

In order to obtain this information one needs further development of QFT in curved space-time.

Conclusions.

- **Integrating conformal anomaly is very efficient, economic and explicit way to derive the EA. It is the best known way to obtain the non-local part of the vacuum EA.**

- **The conformal symmetry can not be exact, it is only a useful approximation (there are differen opinions on this point!). And its effectiveness is mainly restricted to the one-loop level.**

- **There are many generalizations of the original method, including for the theories with other background fields, such as vector, torsion and scalar. One can even obtain some results for the classically non-conformal massive quantum fields, when treating masses as small perturbations.**

- **The main application is, definitely, the Starobinsky model, with a direct link between QFT methods and cosmology. There are certain chances to check our theories experimentally.**