Uses and Abuses of the Coleman-Weinberg Effective Potential

27-Jan-2014

talk based loosely on:
H.Patel, M.J. Ramsey-Musolf, JHEP 1107 (2011), 029

MPI-K Particle and Astroparticle Seminar

Hiren Patel
hiren.patel@mpi-hd.mpg.de

MAX-PLANCK-GESELLSCHAFT

MAX-PLANCK-INSTITUT FÜR KERNPHYSIK

Outline

- Define the Coleman-Weinberg Effective Potential
- Demonstrate **problems** associated with extracting physical quantities.
- **Solve** the problems
- Time permitting, apply solution to, and discuss:
  - Coleman-Weinberg mechanism
  - RG-group improved vacuum stability analysis.

Root problem: Effective potential is gauge dependent.

Coleman-Weinberg Effective Potential

Phenomenological Definition

Energy (expectation value of H) of state $|\Psi\rangle$ suitably localized at $\bar{\phi}$.

$$\int d^3x\, V_{\text{eff}}(\bar{\phi}) = \min_{|\Psi\rangle} \langle\Psi|\hat{H}|\Psi\rangle$$

subject to
$$\langle\Psi|\hat{\phi}|\Psi\rangle = \bar{\phi}$$
$$\langle\Psi|\Psi\rangle = 1$$

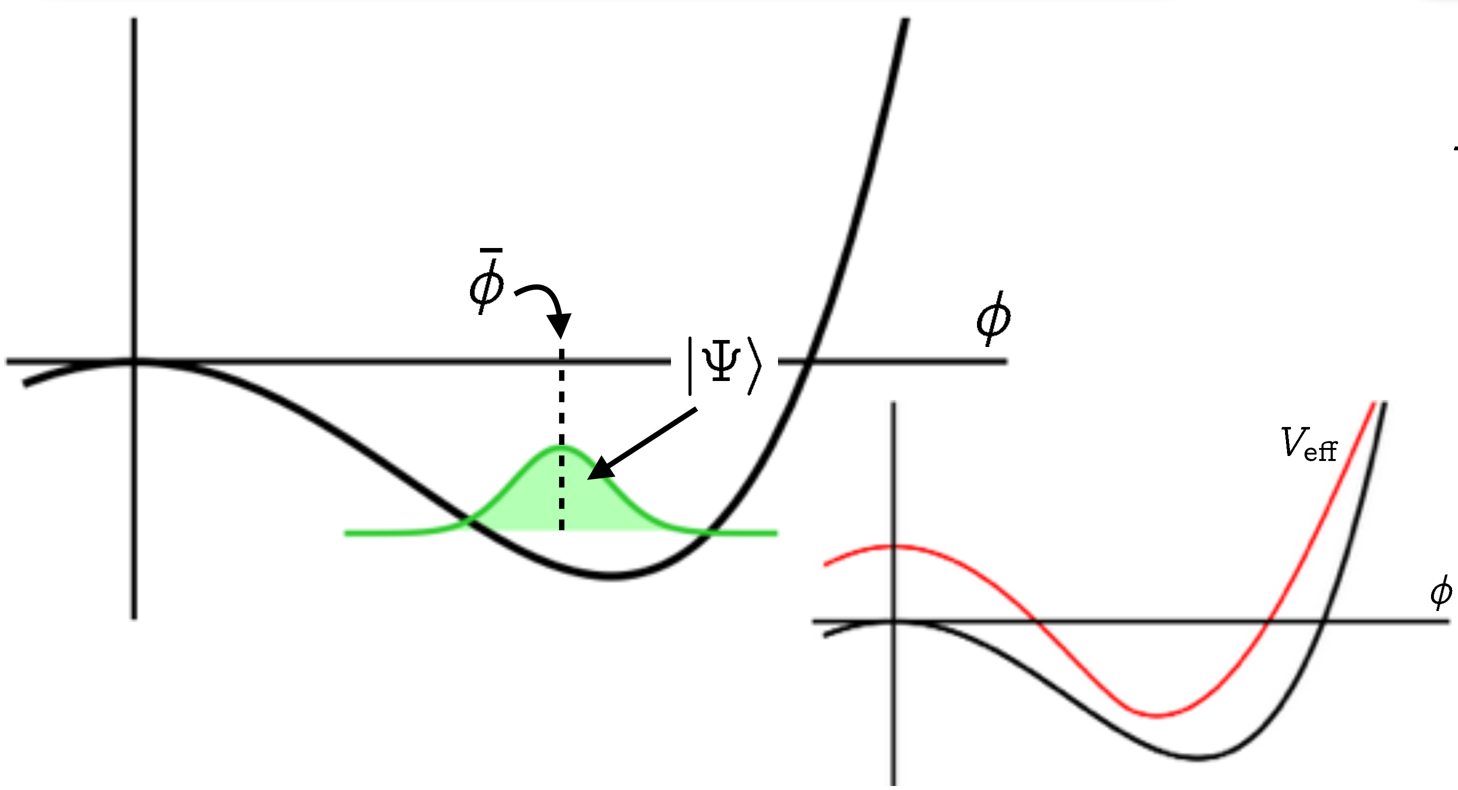
equivalent

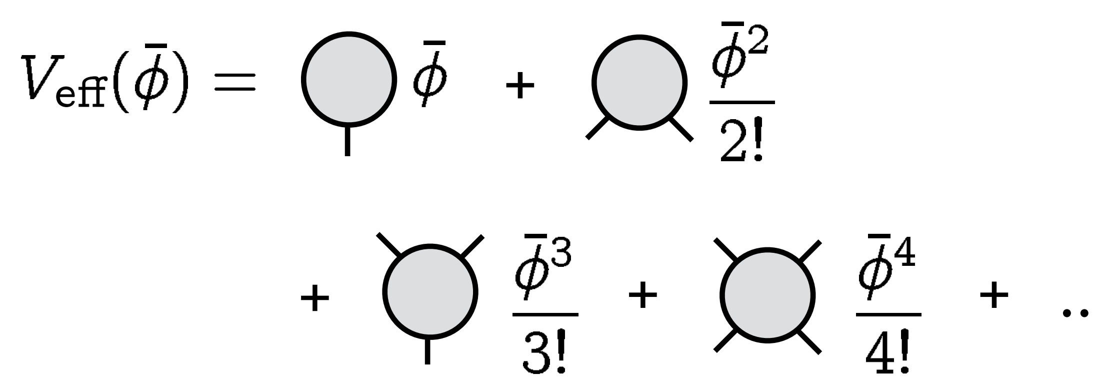
Theoretical Definition

Catalog of 1PI Green's functions at $p = 0$.

$$\int d^4x\, V_{\text{eff}}(\bar{\phi}) = -\Gamma_{\text{eff}}(\bar{\phi}_{(p=0)})$$
$$= -\left(W(j) - \int d^4x\, j\bar{\phi}\right)$$

where j solves $\bar{\phi} = \partial W/\partial j$.





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Coleman-Weinberg Effective Potential

This dual interpretation makes V_{eff} a powerful theoretical tool.

...but is also quite **fragile** when applied to gauge theories.

V_{eff} is **gauge-dependent**,
and harbors **serious gauge artifacts**

and **great care** must be taken in extracting physical quantities.

Applications found in literature:

- Evaluate
  - vacuum energy,
  - condensates (VEVs),
  - masses
- Explore of vacuum structure
- Study dimensional transmutation
- Efficiently calculate (b.g. field method)
  - counterterms
  - correlation functions
- Combine with kinetic terms to study
  - metastable decay, solitons…
- finite T: phase transitions, thermodynamics…

Literature...

It is known that the gauge dependence of the effective potential does not affect the physical quantities [7] ; therefore, we restrict ourselves to the Landau gauge $\alpha = 0$, for simplicity *.

G. Maiella (1979)

effective potential. Note that the effective potential itself is gauge dependent, as noted in Ref. [7(a)]. However, physical quantities obtained from the effective potential, such as the critical temperature, are gauge independent [14].

M. Carrington (1992)

gauge. Although the effective potential itself is gauge-dependent, physical properties following from it, such as its value at stationary points, and the question of whether or not spontaneous symmetry breaking occurs, are gauge invariant

S. Martin (2002)

In fact the effective potential itself is gauge dependent as well, however, all physical quantities derived from the effective potential will be gauge independent. Note there exists a subtlety in

B, Grinstein, M. Trott (2008)

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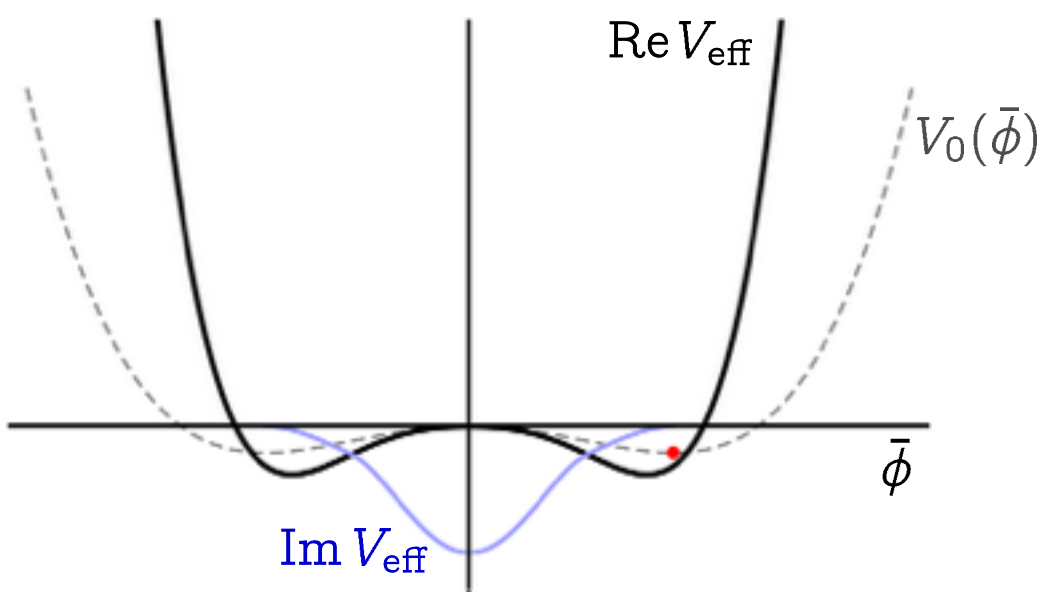
Method employed in literature

Abelian-Higgs model

$$\mathcal{L} = |D_\mu H D^\mu H| - \frac{1}{4}F_{\mu\nu} - V_0(H)$$
$$-\frac{1}{2\xi}\left(\partial.A - \xi e\bar{\phi}h\right)^2 - \bar{\eta}\left(\partial^2 + \xi e^2\bar{\phi}(\bar{\phi}+h)\right)\eta$$

Potential:

$$V_0(H) = -\mu^2|H|^2 + \lambda(|H|^2)^2$$



Step 1: Fix a gauge

Background renormalizable (R_ξ).

$$\mathcal{F} = \partial.A - \xi e\bar{\phi}h = 0$$

(Goldstone modes acquire gauge-dependent mass: $\boxed{m_G^2 = \xi e^2\bar{\phi}^2}$)

Step 2: Compute effective potential in $\xi = 0$ gauge.

Fixed order (# loops) in "perturbation theory".

$$V_{\text{one-loop}} = -\frac{1}{4(4\pi)^2}\sum_i m_i(\bar{\phi})^4 \ln\left(\frac{m_i(\bar{\phi})^2}{\mu_R}\right)$$

But wait! There is an imaginary part!

some $m_i(\bar{\phi})^2 < 0$

Method employed in literature

Abelian-Higgs model

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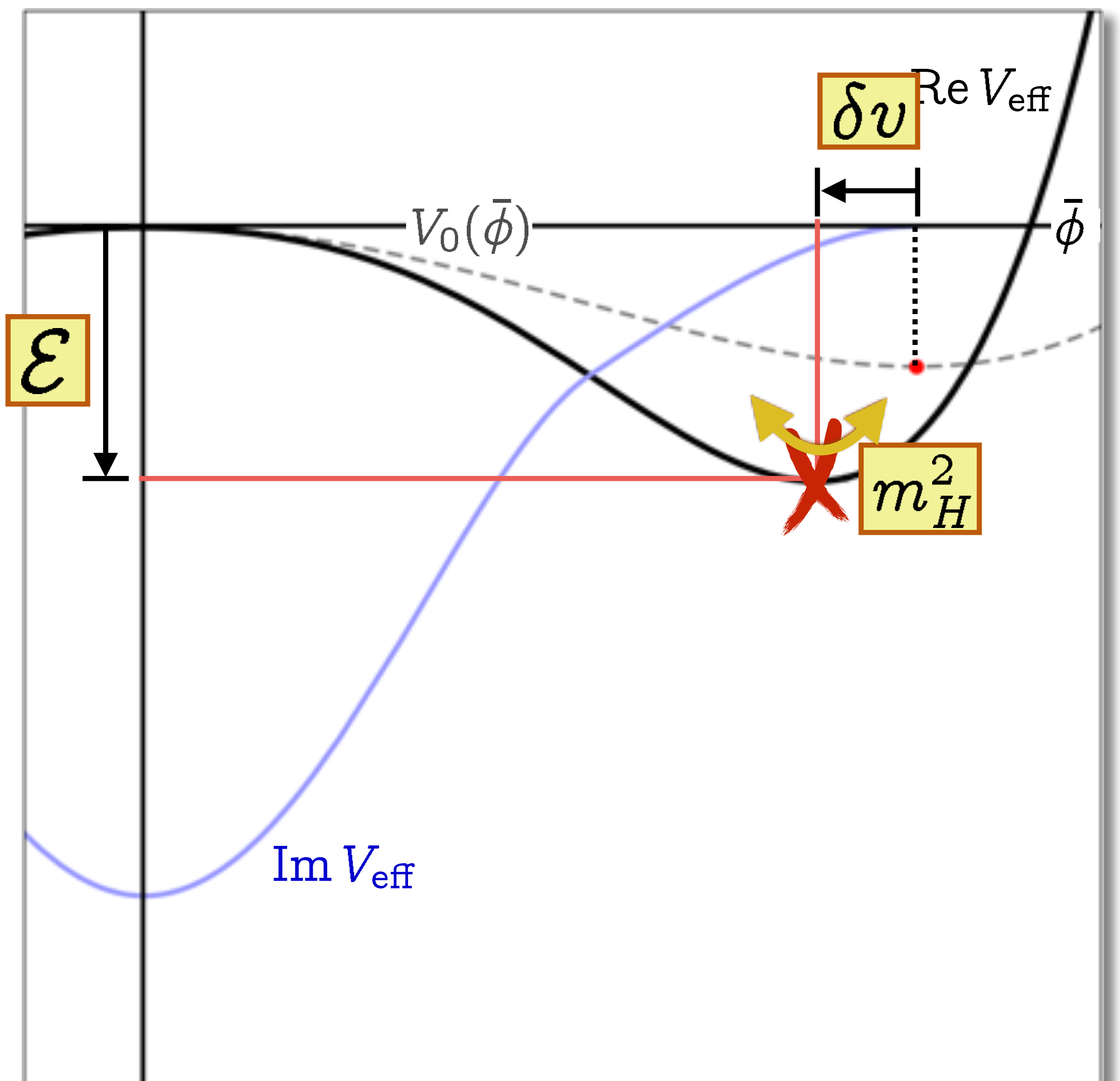


Can't minimize with an imaginary part...

Step 2 ½: "Take real part (for now)"

Step 3: Minimize real part of effective potential.
(by brute force)

Read off the 'physical quantities'

δv = correction to VEV
(usually negative)

$\mathcal{E}$ = vacuum energy

$$\left.\frac{\partial^2 V_{\mathrm{eff}}}{\partial\phi^2}\right|_{\phi_{\min}} \approx m_H^2$$ = Higgs mass-square

Problems

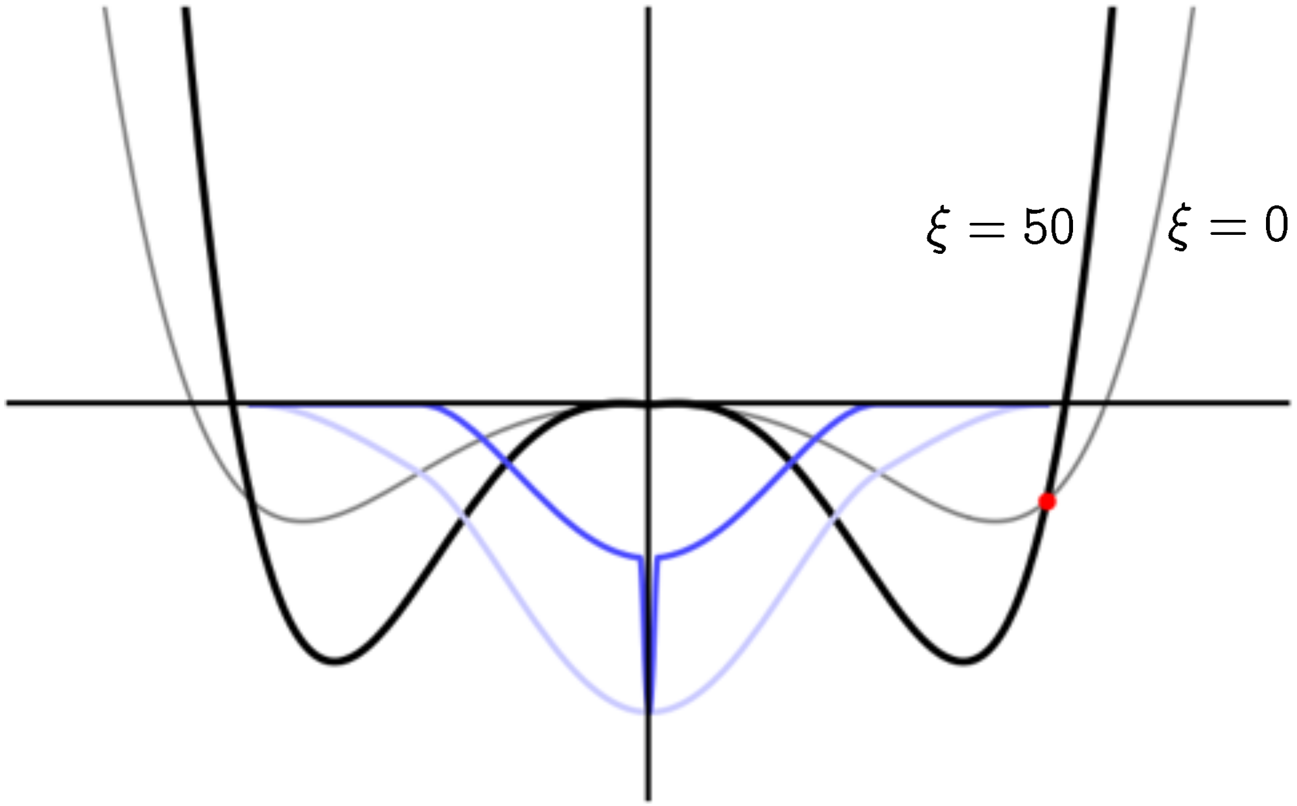
1. For phys. relevant models (incl. SM)

$$\left.\begin{array}{l}\mathcal{E}\\ m_H^2\end{array}\right\}$$ are **complex**!

suggests **instability**:

Not the meta-stability kind, but of the Weinberg-Wu type.

PRD **36**, 2474 (1987)



2. V_{eff} is gauge-dependent, and "physical quantities" inherit the gauge-dependence.

Can make the imaginary part vanish by moving to Feynman gauge (or bigger)

Can make the real part larger (negative) by further raising.

Nielsen's Identities

These statements on the effective potential are not unfounded.

It is known that the gauge dependence of the effective potential does not affect the physical quantities [7]; therefore, we restrict ourselves to the Landau gauge $\alpha = 0$, for simplicity *.

In fact ... however, all physical quantities derived from the effective potential will be gauge independent. Note there exists a subtlety in

gauge. Although the effective potential itself is gauge-dependent, physical properties following from it, such as its value at stationary points, and the question of whether or not spontaneous symmetry breaking occurs, are gauge invariant

Can be traced back to original paper:

Nuclear Physics B101 (1975) 173–188

ON THE GAUGE DEPENDENCE OF SPONTANEOUS SYMMETRY BREAKING IN GAUGE THEORIES

N.K. NIELSEN
Institute of Physics, University of Aarhus, DK 8000 Aarhus C, Denmark

Received 2 September 1975

The Ward-Takahashi identities for scalar electrodynamics in Fermi gauges are to imply a homogeneous first-order partial differential equation for the effective ...

Nielsen's Identity for the Effective Potential:

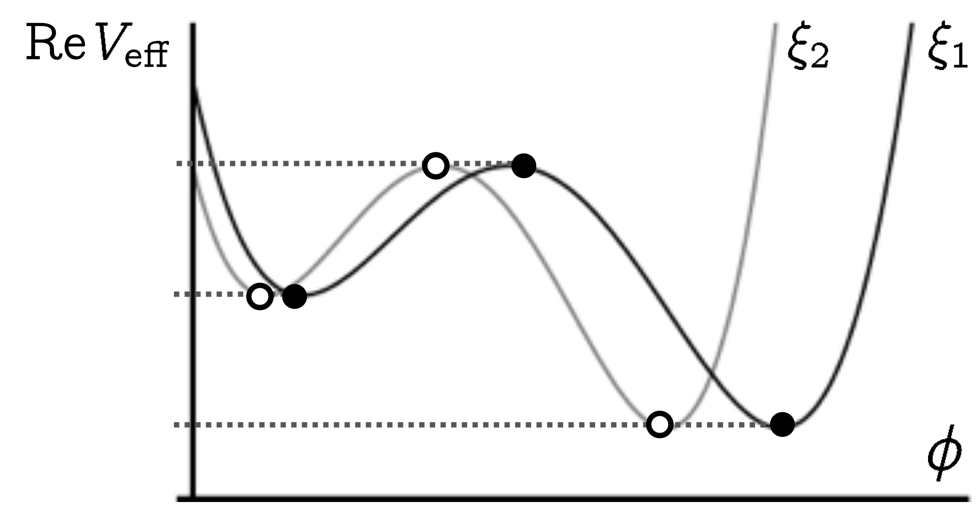
$$\left(\xi \frac{\partial}{\partial \xi} + C(\phi, \xi)\frac{\partial}{\partial \phi}\right)V(\phi, \xi) = 0$$

can be rearranged to:

$$\boxed{\xi \frac{\partial}{\partial \xi} V(\phi, \xi) = -C(\phi, \xi)\frac{\partial}{\partial \phi} V(\phi, \xi)}$$

variations with gauge-parameter... is prop. to ...variations with field.

Specifically, at critical points of V, V is **gauge independent**.



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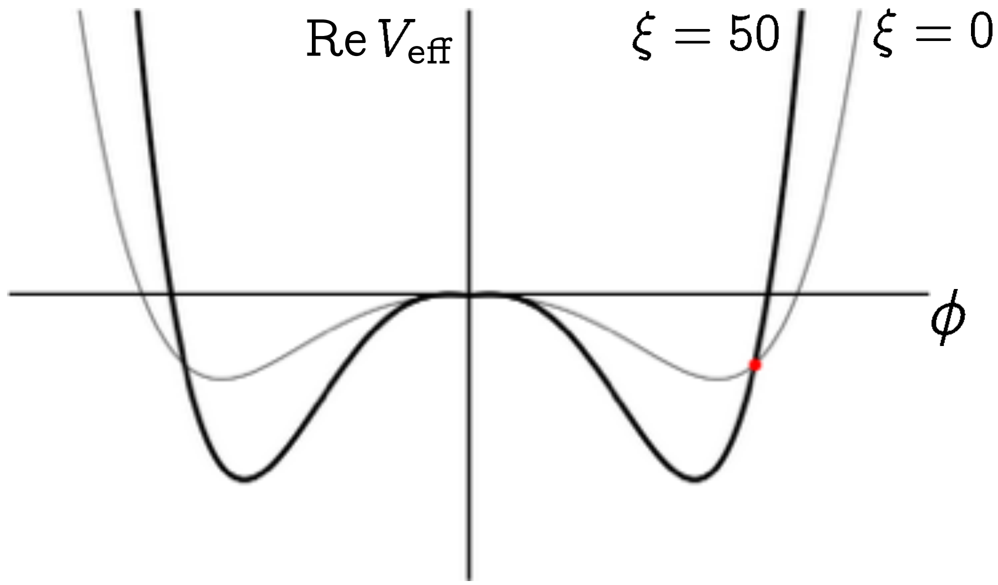
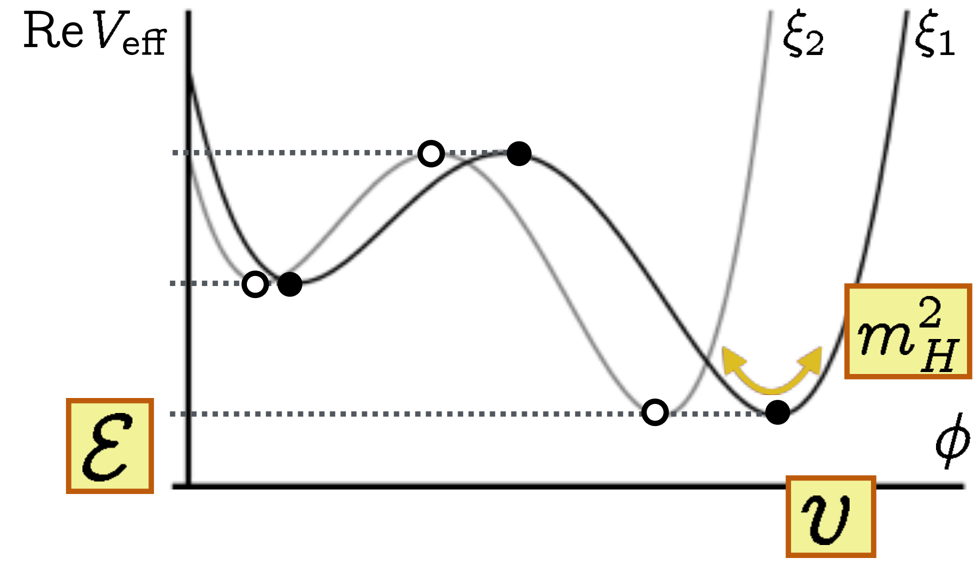
Three Questions:

1. Why does **computed** V_{eff} behave differently than as implied by **Nielsen**?





2. Exactly, which derived quantities are gauge-independent?

		ξ-ind?	
$\mathcal{E}$	vacuum energy	✔	$\langle \Omega \| \hat{H} \| \Omega \rangle$
δv	correction to VEV	✗	$\langle \Omega \| \hat{\Phi} \| \Omega \rangle$
$\left.\frac{\partial^2 V_{\text{eff}}}{\partial \phi^2}\right\|_{\phi_{\text{min}}} \approx m_H^2$	"Higgs mass" -square	✗	exact definition: $p^2 - m_r^2 - \Sigma(p^2)\big\|_{m^2_{\text{pole}}} = 0$

3. And how can we extract them in a gauge-independent manner?

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"$\hbar$—Expansion Method"

H.Patel, M.J. Ramsey-Musolf, JHEP 1107 (2011), 029

originally proposed in context of finite temperature. But works much better at zero-temperature.

Key observation:

"Brute force" minimization retains incomplete higher-order estimates. Must drop these incomplete estimates. $\boxed{\mathcal{E}}$

Calculation of the vacuum energy:

1. Insert $\hbar^{-1}$ here

$$Z[j] = \int \mathcal{D}\Phi\mathcal{D}A\, e^{-\frac{1}{\hbar}(S_E[\Phi]+j\Phi)}$$

($\hbar$ counts # of loops)

2. Effective potential will be a series in $\hbar$.

$$V_{\text{eff}} = V_0 + \hbar V_1 + \hbar^2 V_2 + \ldots$$

as will its derivative: $\partial/\partial\phi$

$$V'_{\text{eff}} = V'_0 + \hbar V'_1 + \hbar^2 V'_2 + \ldots$$

Minimization condition:

$$V'_{\text{eff}}\big|_{\phi_{\min}} = 0$$

3. Solve by "inversion of series":

$$\phi_{\min} = \phi_0 + \underbrace{\hbar\phi_1 + \hbar^2\phi_2 + \ldots}_{\text{quant. corr. } \delta v}$$

"tree-level VEV" $v = \sqrt{\mu^2/\lambda}$

Insert into minimization condition

$$V'_{\text{eff}}(\phi_{\min}) = V'_0(\phi_0 + \hbar\phi_1 + \ldots) + \hbar V'_1(\phi_0 + \hbar\phi_1 + \ldots) + \ldots = 0$$

expand:

$$= V'_0(\phi_0) + \hbar\left[V'_1(\phi_0) + \phi_1 V''_0(\phi_0)\right] = 0$$

Each power of h-bar must satisfy equality.

$\mathcal{O}(1):\ V'_0(\phi_0) = 0$ establishes "tree-level VEV"

$\mathcal{O}(\hbar):\ \phi_1 = -V''_0(\phi_0)^{-1} V'_1(\phi_0)$ **GENUINE** one-loop correction

$\vdots$

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originally proposed in context of finite temperature. But works much better at zero-temperature.

Key observation:

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Calculation of the vacuum energy:

$$\phi_{\min} = \phi_0 + \hbar\phi_1 + \hbar^2\phi_2 + \ldots$$

$$\mathcal{O}(1): \; V_0'(\phi_0) = 0$$

$$\mathcal{O}(\hbar): \; \phi_1 = -V_0''(\phi_0)^{-1}V_1'(\phi_0)$$

4. Insert into potential:

$$\begin{aligned}
\mathcal{E} &= V_{\text{eff}}(\phi_{\min}) \\
&= V_0(\phi_{\min}) + \hbar V_1(\phi_{\min}) + \hbar^2 V_2(\phi_{\min}) + \ldots \\
&= V_0(\phi_0+\hbar\phi_1+\ldots) + \hbar V_1(\phi_0+\hbar\phi_1+\ldots) + \hbar^2 V_2(\phi_0+\hbar\phi_1+\ldots)+\ldots
\end{aligned}$$

expand once more

$$\begin{aligned}
&= V_0(\phi_0) + \hbar\Big(V_1(\phi_0) + \phi_1 \cancel{V_0'(\phi_0)}\Big) \\
&\quad +\hbar^2\Big(V_2(\phi_0) + \phi_1 V_1'(\phi_0) + \frac{1}{2}\phi_1^2 V_0''(\phi_0) + \phi_2 \cancel{V_0'(\phi_0)}\Big)
\end{aligned}$$

Therefore, to order $\hbar$, the vacuum energy is obtained by evaluating the one-loop potential at the tree-level minimum ϕ_0.

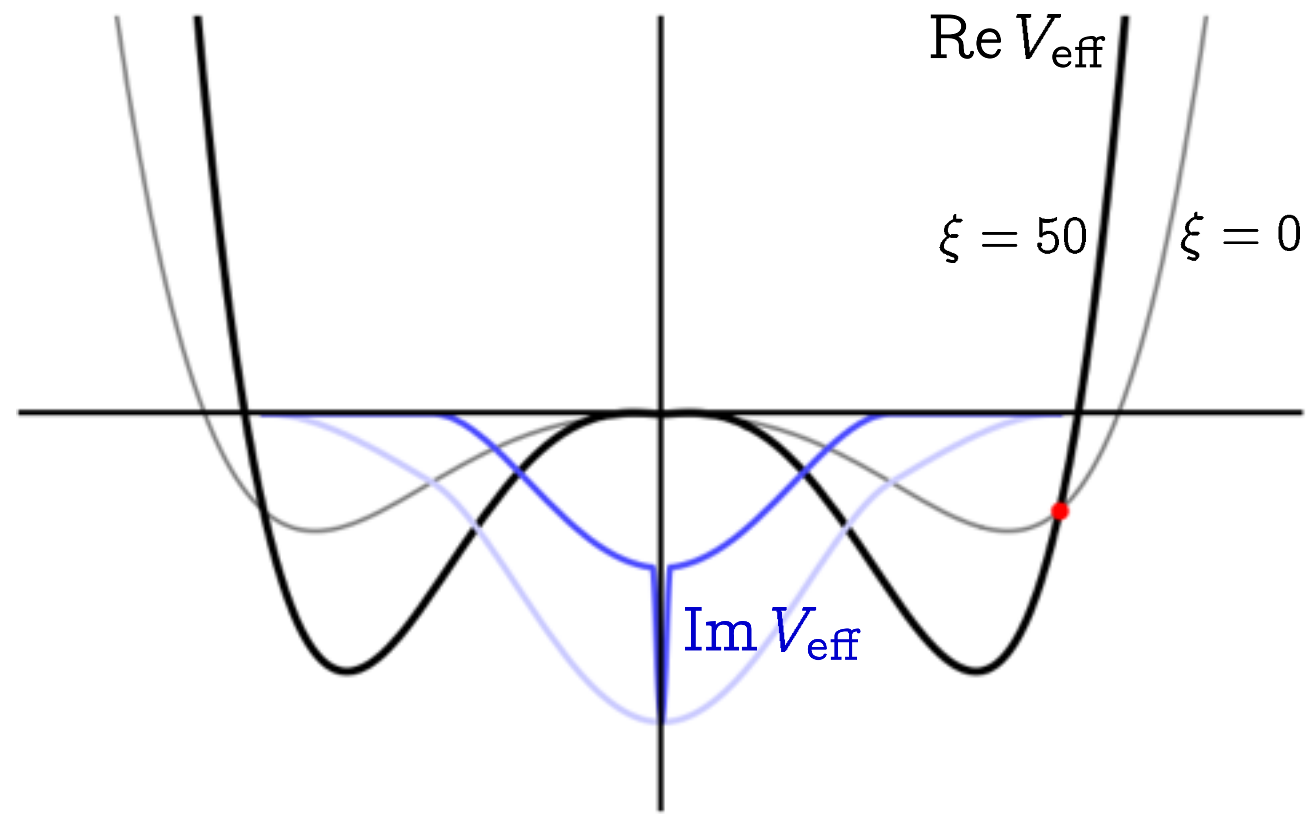
$$\boxed{\begin{aligned}\mathcal{E} &= V_0(\phi_0) + \hbar V_1(\phi_0) \\ &\quad + \hbar^2\left(V_2(\phi_0) - \frac{1}{2}\frac{V_1'(\phi_0)^2}{V_0''(\phi_0)}\right) + \ldots\end{aligned}}$$

"$\hbar$—Expansion Method"

$$\mathcal{E} = V_0(\phi_0) + \hbar V_1(\phi_0) + \hbar^2\left(V_2(\phi_0) - \frac{1}{2}\frac{V_1'(\phi_0)^2}{V_0''(\phi_0)}\right) + \ldots$$

Theoretical Properties

- Manifestly gauge-independent ✔
- No spurious imaginary part in broken phase
- Valid for all extrema (incl. maxima and saddle)



Practical advantage

Difficult numerics associated with minimization in a complicated, multi-loop function is gone.

All that is needed is to find the extrema of the tree-level potential (easy), and the series generates the corrections.

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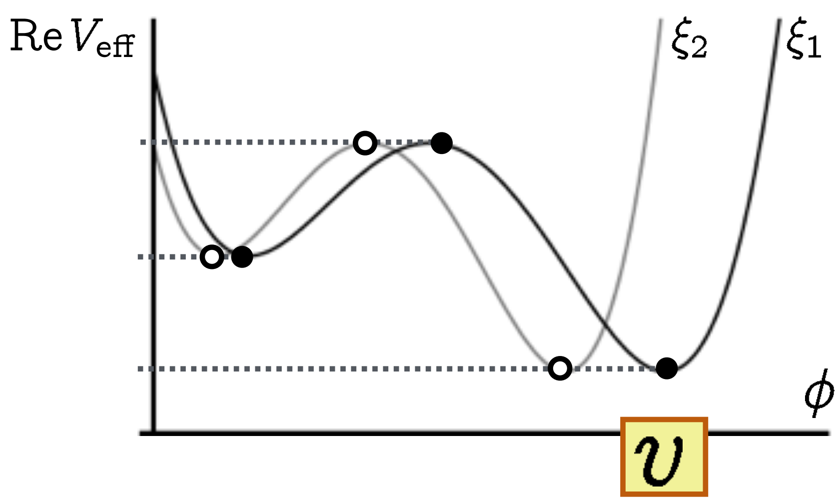
Vacuum expectation value

Nielsen id.

$$\xi \frac{\partial}{\partial \xi} V(\phi, \xi) = -C(\phi, \xi) \frac{\partial}{\partial \phi} V(\phi, \xi) \quad \Longrightarrow \quad \frac{\partial \phi_{\min}}{\partial \xi} = C(\phi_{\min}, \xi) \neq 0$$

VEV not gauge-independent! ✗



Reason:

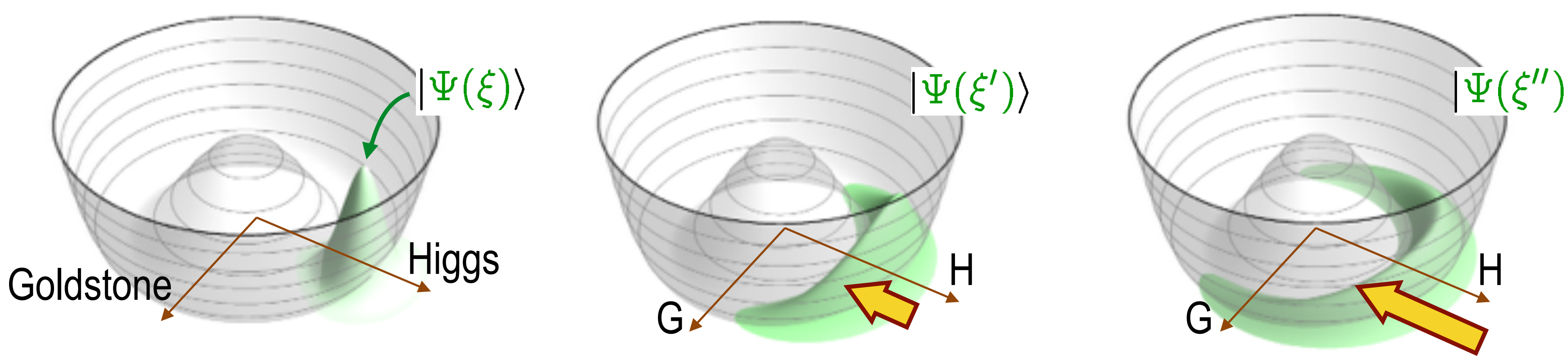
$$\phi_{\min} = \langle \Psi_{(\xi)} | \hat{\Phi} | \Psi_{(\xi)} \rangle$$

$$\hat{\Phi} \to \hat{\Phi}\, e^{i\alpha(x).T}$$

field-op. not invariant!

My Interpretation:

Ground state changes as ξ is varied. M.E. gives mean of projection.



Need 'radial' field-op. for ξ-ind. VEV.

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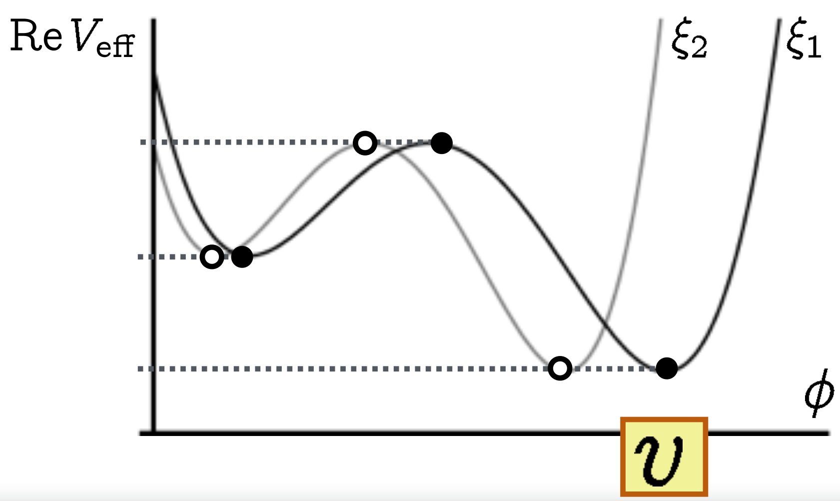
Vacuum expectation value

Nielsen id.

$$\xi \frac{\partial}{\partial \xi} V(\phi, \xi) = -C(\phi, \xi) \frac{\partial}{\partial \phi} V(\phi, \xi) \quad \Longrightarrow \quad \frac{\partial \phi_{\rm min}}{\partial \xi} = C(\phi_{\rm min}, \xi) \neq 0$$

VEV not gauge-independent! ✗



Reason:

$$\phi_{\rm min} = \langle \Psi_{(\xi)} | \hat{\Phi} | \Psi_{(\xi)} \rangle$$

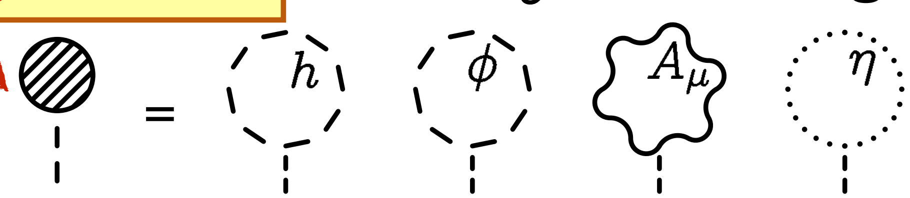
$$\hat{\Phi} \to \hat{\Phi}\, e^{i\alpha(x).T}$$

field-op. not invariant!

$\hbar$-expansion method:

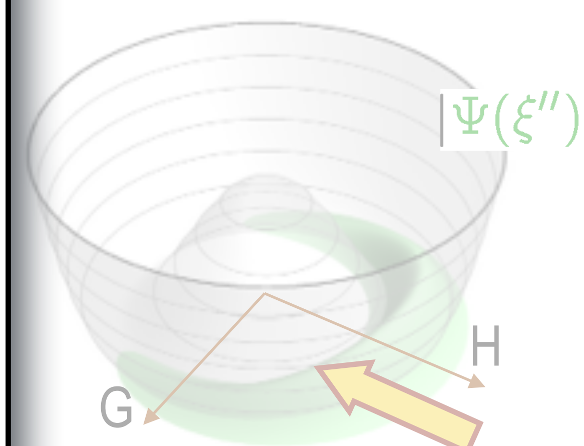
$$\boxed{\begin{aligned} \phi_{\rm min} &= \phi_0 + \hbar \phi_1 + \hbar^2 \phi_2 + \dots \\ \mathcal{O}(\hbar): \quad \phi_1 &= -V_0''(\phi_0)^{-1} V_1'(\phi_0) \end{aligned}}$$

1. agreement with Feynman diagrams.



2. Always real-valued.

gives mean of projection.



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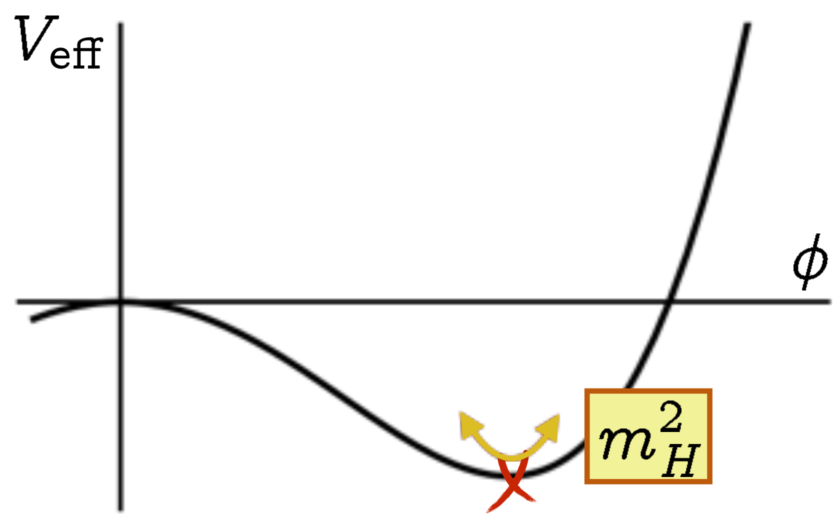
Higgs Masses



"Effective Potential Approximation"

$$p^2 - m_r^2 - \Sigma(p^2)\Big|_{p^2=M^2_{\rm pole}} = 0$$

$$M^2_{\rm pole} - \underbrace{\left(m_r^2 + \boxed{\Sigma(0)}\right)} + \Sigma(0) - \Sigma(M^2_{\rm pole}) = 0$$

$$M^2_{\rm pole} = \frac{\partial^2 V_{\rm eff}}{\partial \phi^2}\Big|_{\phi_{\rm min}} - \cancel{\left(\Sigma(0) - \Sigma(M^2_{\rm pole})\right)}$$ small?

Practical Advantage:

$V_{\rm eff}$ much easier to compute than self-energy graphs.

— used often in SUSY models

R.J. Zhang Phys. Lett B 447, 89 (1999)

Espinosa, Zhang, JHEP 03, 026 (2000)

S. Martin, Phys Rev D 67, 095012 (2003)

Theoretical Disadvantages:

1. Yields imaginary masses

2. $\Sigma(0)$ is not gauge-invariant!

$\therefore$ $\frac{\partial^2 V_{\rm eff}}{\partial \phi^2}\Big|_{\phi_{\rm min}}$ not gauge-invariant; even if $\hbar$ -expansion is used.

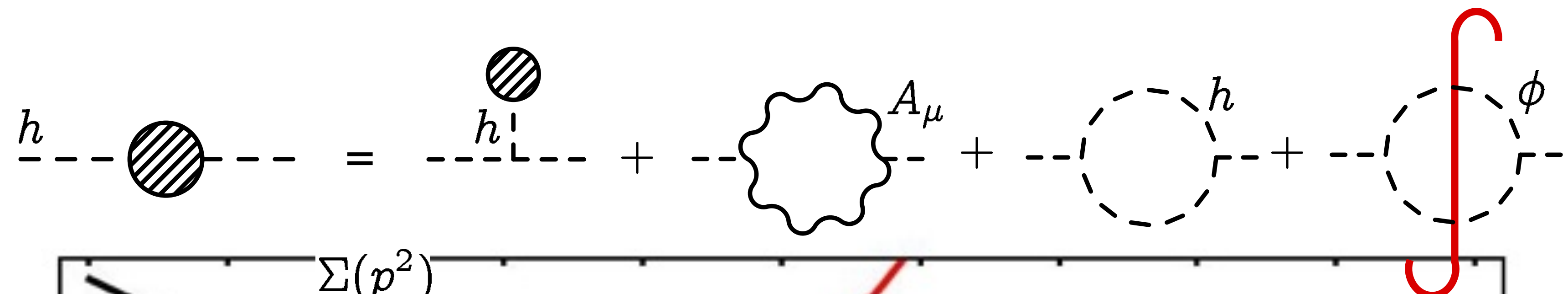
(serious)

3. size of error?

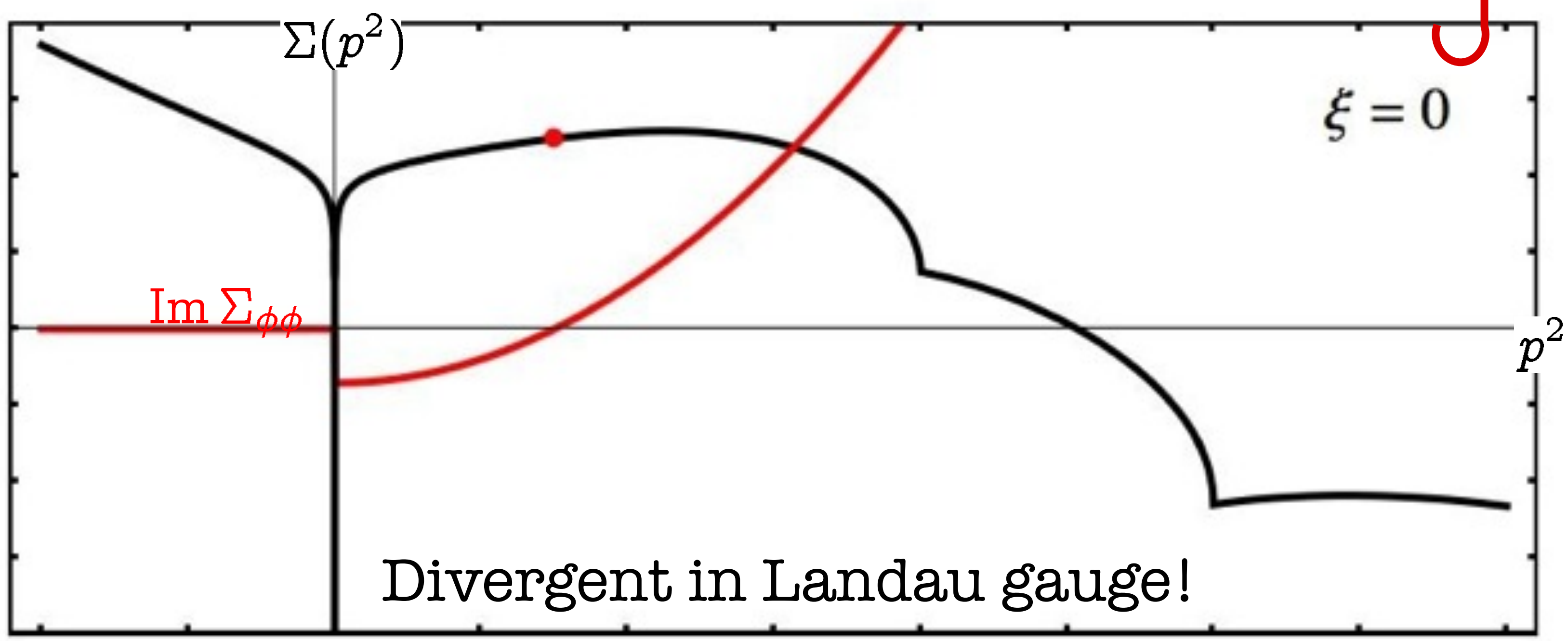
a little secret...

$$M^2_{\rm pole} = \overbrace{\frac{\partial^2 V_{\rm eff}}{\partial\phi^2}\bigg|_{\phi_{\rm min}}}^{\xi=0} - \overbrace{\Big(\Sigma(0) - \Sigma(M^2_{\rm pole})\Big)}^{\xi=1}$$

$\Sigma(p^2) =$



$+$ 5 others

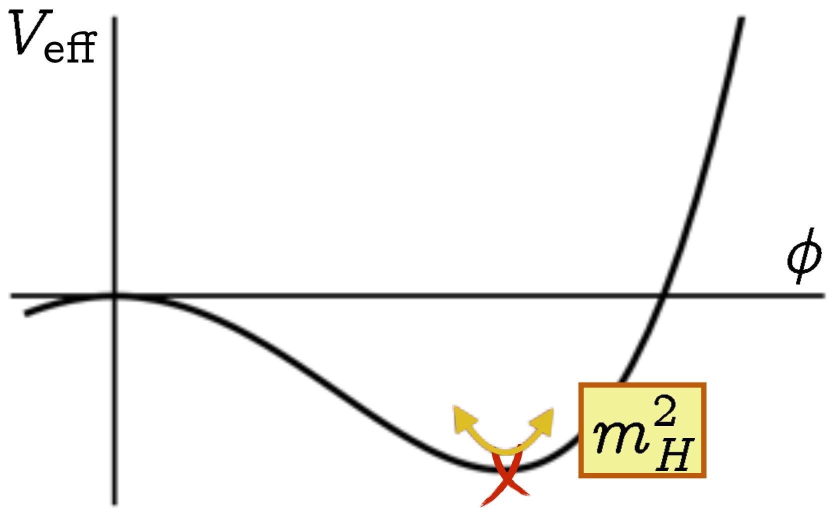


$$\Sigma(p^2=0) = \frac{\hbar\mu^2}{(4\pi)^2}\left[\frac{6e^4}{\lambda} + 12\lambda + 2\xi e^2 + 6\lambda\ln\left(\frac{2\mu^2}{\mu_R^2}\right) + 2(\lambda-\xi e^2)\ln\left(\frac{\xi e^2\mu^2}{\lambda\mu_R^2}\right)\right]$$

Only revealed by $\hbar$-expansion.

$$\frac{\partial^2 V_{\rm eff}}{\partial\phi^2}\bigg|_{\phi_{\rm min}} = V_0''(\phi_0) + \hbar\Big(V_1''(\phi_0) - V_0''(\phi_0)^{-1}V_1'(\phi_0)V_0'''(\phi_0)\Big) + \ldots$$

Higgs Masses



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$\therefore$ $\frac{\partial^2 V_{\text{eff}}}{\partial\phi^2}\Big|_{\phi_{\text{min}}}$ not gauge-invariant; even if $\hbar$-expansion is used.

(serious)

3. 'Hidden' divergence in $\xi = 0$ gauge! Only revealed by $\hbar$-expansion.

Summary

ξ-ind?

$\boxed{\mathcal{E}}$ vacuum energy ✔ $\langle\Omega|\hat{H}|\Omega\rangle$

Gauge-independent only following $\hbar$ -expansion.

$\boxed{\delta v}$ correction to VEV ✗ $\langle\Omega|\hat{\Phi}|\Omega\rangle$ matrix elements of non-gauge-invariant operators are not expected to be gauge-independent.

Matches Feynman pert. th. only following $\hbar$-expansion.

$\boxed{m_H^2}$ "Higgs mass" -square ✗

exact definition:

$$p^2 - m_{\rm r}^2 - \Sigma(p^2)\Big|_{m^2_{\rm pole}} = 0$$

$\partial^2 V/\partial h^2$ is approximating $\Sigma(0)$, not $m^2_{\rm pole}$.
Further, $\Sigma(0)$ is gauge-dependent, and singular when $\xi = 0$.

Theoretical **abomination**.
Never use.

Application to more advanced settings?

1. Coleman-Weinberg Mechanism
2. RG-improved vacuum stability analysis
3. Metastability analysis
4. Phase transition analysis

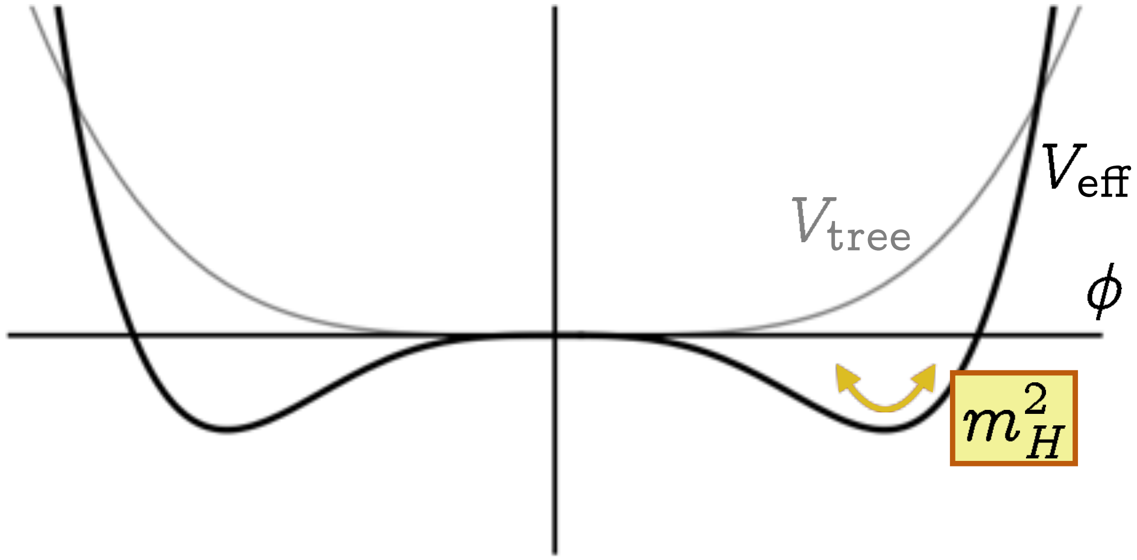
Application #1: Coleman-Weinberg mech.

Paradigm:

Start with $V(\Phi) = 0$ or $= \lambda|\Phi|^4$ small.
(no mass scale)

quantum effects generate mass scale.

$\boxed{v}, \boxed{\mathcal{E}}, \boxed{m_H^2}, \boxed{m_A^2} \neq 0$



It is possible to analyze this case in a gauge-invariant fashion by using the $\hbar$-expansion method.

Insert $\hbar^{-1}$ here... ...and $\hbar$ here

$$Z[j] = \int \mathcal{D}\Phi \mathcal{D}A \, e^{-\frac{1}{\hbar}(S_E[\Phi]+j\Phi)} \qquad V(\Phi) = \hbar\lambda|\Phi|^4$$

($\hbar$ no longer counts # of loops)

In this case,

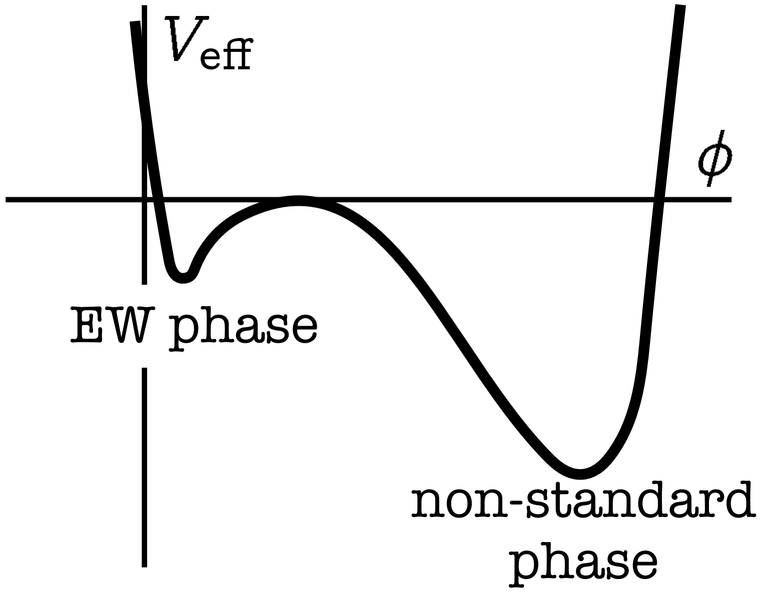
$$V_{\text{eff}} = 0 + \hbar V_1 + \ldots$$

both tree- and **part** of one-loop

and, all other aspects of analysis goes through.

Application #2: RG-improved vacua

It is well known that in the Standard Model, there is a 'non-standard' vacuum lying at extremely large field values:



This is revealed only by an 'RG-improvement' of the effective potential,

Callan-Symanzik eqn:

$$\left(\beta_e \frac{\partial}{\partial e} + \beta_\lambda \frac{\partial}{\partial \lambda} - \gamma\phi \frac{\partial}{\partial \phi} + \mu \frac{\partial}{\partial \mu}\right) V_{\text{eff}}(\phi, \mu) = 0$$

This improvement entails a re-summation of logarithms **across all orders** in the loop expansion.

This is **incompatible** with the $\hbar$-expansion method needed to maintain gauge-independence.

Nielsen identity:

$$\xi \frac{\partial}{\partial \xi} V_{\text{eff}} = -C(\phi, \xi) \frac{\partial}{\partial \phi} V_{\text{eff}}(\phi, \xi)$$

A simple solution exists provided the R.G. coefficient functions are gauge-independent.

$$\beta_e = \frac{1}{(4\pi^2)} \frac{e^3}{3}$$

$$\beta_\lambda = \frac{1}{(4\pi^2)} \left(20\lambda^2 - 12e^2\lambda + 6e^4\right)$$

$$\gamma = -\frac{1}{(4\pi)^2}(3 + \xi)e^2$$

Unfortunately, anomalous dimensions are gauge-dependent. Still an open research problem.

Thank you!