Symmetry breaking in non-Hermitian, PT-symmetric quantum field theories

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Based on work with J. Alexandre, C. M. Bender, J. Ellis and D. Seynaeve

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Outline

- Introduction and motivation
- Toy scalar model
- Variational procedure
- Exceptional points
- Continuous symmetries: Noether's theorem, the Goldstone theorem, gauge symmetry and the Higgs mechanism
  (Alexandre, PM & Seynaeve '17; Alexandre, Ellis, PM & Seynaeve '18 & '19; Mannheim '19; Fring & Taira '19)
- Concluding remarks

$\mathcal{PT}$-symmetry versus Hermiticity

All **Hermitian matrices** have **real eigenvalues**, but not all matrices with real eigenvalues are Hermitian.

So which should we be motivated by:

- the **Hermiticity** of the matrix representation?
- or the **reality** of observables?

The **reality** of observables can instead be guaranteed by the **weaker condition** of **unbroken** $\mathcal{PT}$ **symmetry**. (see Bender & Boettcher '98)

And **unitarity**, by virtue of the existence of the $\mathcal{C}'$ operator. (see Bender, Brody & Jones '02)

In fact, existence of an **antilinear discrete symmetry** of the Hamiltonian is sufficient. (see Mannheim '18)

A few examples

- Photonics (see, e.g., Longhi ‘10 & ‘17; El-Ganainy et al. ‘17)
- Critical phenomena (see, e.g., Ashida, Furukawa & Ueda ‘17)
- $i\phi^3$ theory (see Blencowe, Jones & Korte ‘98; Bender, Brody & Jones ‘04; Jones ‘04; Bender, Branchina & Messina ‘13; Bender, Hook, Mavromatos & Sarkar ‘16; Shalaby ‘17)
- $-\phi^4$ theory (see Shalaby & Al-Thoyaib ‘10)
- Lattice fermion models (see Chernodub ‘17)
- Dark matter (see Rodionov & Mandel ‘19)
- Higgs-boson decays (see Korchin & Kovalchuk ‘16)
- Deconfinement (see Raval & Mandal ‘18)
- Non-Hermitian chiral magnetic effect (see Chernodub & Cortijo ‘19)
- And many more . . .

A scalar QFT

Consider a scalar theory with a **non-Hermitian mass term**:
(Alexandre, PM & Seynaeve '17)

$$\mathcal{L} = \partial_\alpha \phi_1^* \partial^\alpha \phi_1 + \partial_\alpha \phi_2^* \partial^\alpha \phi_2 - m_1^2 |\phi_1|^2 - m_2^2 |\phi_2|^2 - \mu^2 (\phi_1^* \phi_2 - \phi_2^* \phi_1)$$

$\mathcal{PT}$ symmetric under (for c-number fields)

$$\begin{aligned} \mathcal{P} : \quad & \phi_1(x) \to +\phi_1(x) \qquad \phi_2(x) \to -\phi_2(x) \\ \mathcal{T} : \quad & \phi_1(x) \to +\phi_1^*(x) \qquad \phi_2(x) \to +\phi_2^*(x) \end{aligned}$$

The **mass spectrum**

$$M_\pm^2 = \frac{m_1^2 + m_2^2}{2} \pm \left[\left(\frac{m_1^2 - m_2^2}{2} \right)^2 - \mu^4 \right]^{1/2}$$

is **real**, so long as we are in the unbroken regime of $\mathcal{PT}$ symmetry

$$\eta \equiv \frac{2|\mu^2|}{|m_1^2 - m_2^2|} < 1$$

We have an **exceptional point** at $\eta = 1$.

Variational procedure

Since the action is not Hermitian

$$\frac{\delta S}{\delta \phi_i^*} \equiv \frac{\partial \mathcal{L}}{\partial \phi_i^*} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi_i^*} = 0 \qquad \not\Leftrightarrow \qquad \frac{\delta S}{\delta \phi_i} \equiv \frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi_i} = 0$$

This is just a consequence of the fact that non-Hermitian matrices have distinct **left** and **right eigenvectors**, i.e. the left and right zero modes are distinct.

The two choices are **physically equivalent**, since the difference can be absorbed by a field redefinition.

The action is $\mathcal{PT}$ symmetric though, and we can choose

$$\frac{\delta S}{\delta \phi_i^{\mathcal{PT}}} = 0 \qquad \Leftrightarrow \qquad \left(\frac{\delta S}{\delta \phi_i}\right)^{\mathcal{PT}} = 0$$

or

$$\frac{\delta S}{\delta \phi_i} \;=\; 0 \qquad \Leftrightarrow \qquad \left(\frac{\delta S}{\delta \phi_i^{\mathcal{PT}}}\right)^{\mathcal{PT}} \;=\; 0$$

The left and right eigenvectors are related by $\mathcal{PT}$.

$\mathcal{PT}$ (indefinite) inner product

The mass matrix

$$\begin{pmatrix} m_1^2 & \mu^2 \\ -\mu^2 & m_2^2 \end{pmatrix}$$

has **right eigenvectors**

$$\mathbf{e}_+ = \mathcal{N}\begin{pmatrix} \eta \\ \sqrt{1-\eta^2}-1 \end{pmatrix} \qquad \mathbf{e}_- = \mathcal{N}\begin{pmatrix} 1-\sqrt{1-\eta^2} \\ -\eta \end{pmatrix}$$

and **left eigenvectors**

$$\mathbf{e}_+^\ddagger \equiv \left[\mathbf{e}_+^{\mathcal{PT}}\right]^\mathsf{T} = \mathcal{N}\begin{pmatrix} \eta & 1-\sqrt{1-\eta^2} \end{pmatrix} \qquad \mathbf{e}_-^\ddagger = \mathcal{N}\begin{pmatrix} 1-\sqrt{1-\eta^2} & \eta \end{pmatrix}$$

The eigenvectors are **not** orthogonal with respect to Hermitian conjugation

$$\mathbf{e}_+^\dagger \mathbf{e}_- = 2\mathcal{N}^2\eta\left(1-\sqrt{1-\eta^2}\right)$$

except at the Hermitian point $\mu = 0$.

They are orthogonal with respect to the $\mathcal{PT}$ inner product

$$\mathbf{e}_+^\ddagger \mathbf{e}_- = 0$$

Exceptional point

Normalising with respect to the $\mathcal{PT}$ inner product yields

$$\mathcal{N} = \left(2\eta^2 - 2 + 2\sqrt{1-\eta^2}\right)^{-1/2}$$

$\mathcal{N}$ diverges when $\eta \to 1$, the mass matrix becomes **defective** with **Jordan normal form**

$$\begin{pmatrix} \frac{m_1^2+m_2^2}{2} & 1 \\ 0 & \frac{m_1^2+m_2^2}{2} \end{pmatrix}$$

The two eigenvectors merge

$$\mathbf{e}_+ = \mathbf{e}_- \propto \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The issue with orthogonality is now moot, and we can normalise with respect to the Hermitian inner product, giving $\mathcal{N} = 1/\sqrt{2}$.

On-shell this looks like a Hermitian theory with half the degrees of freedom: the equation of motion is

$$\Box\left(\Phi_1 + \Phi_2\right) + \frac{m_1^2 + m_2^2}{2}\left(\Phi_1 + \Phi_2\right) = 0$$

Noether's theorem

- The Hermitian-conjugate Euler-Lagrange equations **cannot** be satisfied simultaneously.

- Hence, conserved currents **cannot** correspond to symmetries that leave the Lagrangian invariant.

- Instead, conserved currents correspond to transformations that effect a particular variation in the non-Hermitian part of the Lagrangian (Alexandre, PM & Seynaeve '17).

Choosing $\delta S/\delta\Phi^{\ddagger} = 0$, it follows from

$$\delta\mathcal{L} = \left(\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\alpha\frac{\partial\mathcal{L}}{\partial\partial_\alpha\Phi}\right)\delta\Phi + \delta\Phi^{\ddagger}\left(\frac{\partial\mathcal{L}}{\partial\Phi^{\ddagger}} - \partial_\alpha\frac{\partial\mathcal{L}}{\partial\partial_\alpha\Phi^{\ddagger}}\right) + \partial_\alpha j_\delta^\alpha$$

that we have a conserved current if the transformation is such that

$$\delta\mathcal{L} = \left(\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\alpha\frac{\partial\mathcal{L}}{\partial\partial_\alpha\Phi}\right)\delta\Phi$$

$U(1)$ current

The conserved current is

$$j^\alpha = i[\phi_1^* \partial^\alpha \phi_1 - (\partial^\alpha \phi_1^*)\phi_1] - i[\phi_2^* \partial^\alpha \phi_2 - (\partial^\alpha \phi_2^*)\phi_2]$$

corresponding to

$$\Phi \to e^{i\theta P}\Phi = \begin{pmatrix} e^{+i\theta}\phi_1 \\ e^{-i\theta}\phi_2 \end{pmatrix} \qquad P \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Notice that this reflects the usual interpretation of a $\mathcal{PT}$ symmetric system in terms of one with **balanced gain** and **loss**.

This leads to a family of equivalent non-Hermitian theories:

$$\mathcal{L} = \partial_\alpha \phi_1^* \partial^\alpha \phi_1 + \partial_\alpha \phi_2^* \partial^\alpha \phi_2 - m_1^2|\phi_1|^2 - m_2^2|\phi_2|^2 - \mu^2(e^{-2i\theta}\phi_1^*\phi_2 - e^{+2i\theta}\phi_2^*\phi_1)$$

with

$$\delta\mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha \Phi}\right)\delta\Phi = 2i\theta\mu^2(\phi_1^*\phi_2 + \phi_2^*\phi_1)$$

The **eigenspectrum** is invariant under the global $U(1)$ transformation.

A fermionic example

Non-Hermitian extension of the **Dirac theory**: (Bender, Jones & Rivers '05; Alexandre, Bender & PM '15; Alexandre, Bender & PM '15; Alexandre, PM & Seynaeve '17)

$$\mathcal{L} = \bar{\psi}(x)\big(i\gamma^\alpha \partial_\alpha - m - \mu\gamma^5\big)\psi(x)$$

The **conserved current** is

$$j_-^\alpha = \bar{\psi}\gamma^\alpha\left(1 + \frac{\mu}{m}\gamma^5\right)\psi = \psi_L^\dagger \bar{\sigma}^\alpha \psi_L \left(1 - \frac{\mu}{m}\right) + \psi_R^\dagger \sigma^\alpha \psi_R \left(1 + \frac{\mu}{m}\right)$$

corresponding to

$$\psi \to \psi' = \exp\left[+i\theta\left(1 + \frac{\mu}{m}\gamma^5\right)\right]\psi \qquad \bar{\psi} \to \bar{\psi}' = \bar{\psi}\exp\left[-i\theta\left(1 - \frac{\mu}{m}\gamma^5\right)\right]$$

and giving

$$\delta\mathcal{L} = -2\mu\bar{\psi}\gamma^5\delta\psi \neq 0$$

For $\mu \to +(-)\, m$, the left(right)-chiral current decouples, and the squared mass eigenvalues $M^2 = m^2 - \mu^2$ go to zero.

A fermionic example

Gauging the model, the Lagrangian at the exceptional point $\mu = +m$ is

$$\mathcal{L} = \psi_L^\dagger i\bar{\sigma} \cdot D_- \psi_L + \psi_R^\dagger i\sigma \cdot D_+ - 2m\psi_L^\dagger \psi_R \qquad D_\pm^\alpha = \partial^\alpha + i\left(g_V \pm g_A\right) A^\alpha$$

with equations of motion

$$i\sigma \cdot D_+ \psi_R = 0 \qquad i\bar{\sigma} \cdot D_- \psi_L = 2m\psi_R$$

We can integrate out the left-chiral field to obtain

$$\mathcal{L}_{\mathrm{on-shell}} = \psi_R^\dagger i\sigma \cdot D_+ \psi_R \tag{1}$$

But this is the Hermitian theory of a single, massless right-handed Weyl fermion!

And the axial U(1) gauge symmetry is restored:

$$p_\alpha \Pi^{\alpha\beta}(p) = \frac{g_A^2}{\pi^2}\, p^\beta \left(m^2 - \mu^2\right) B_0 \underset{\mu \to \pm m}{\longrightarrow} 0$$

Massless Dirac fermions can undergo flavour oscillations (Jones-Smith & Mathur '14).

A Higgs-Yukawa theory

We can realize a **non-Hermitian Yukawa theory of light neutrinos** (Bender, Alexandre & PM '15):

$$\mathcal{L} = \bar{L}_L i\not{D} L_L + \bar{\nu}_R i\not{\partial}\nu_R - h_- \bar{L}_L \tilde{\phi}\nu_R - h_+ \bar{\nu}_R \tilde{\phi}^\dagger L_L \qquad h_\pm = h \pm \eta$$

After SSB, we have a squared mass

$$M^2 = \frac{v^2}{2}(h^2 - \eta^2)$$

We can promote h and η to matrices in flavour space:

$$M^2_{1(2)} = \frac{v^2}{4}\left[\mathrm{tr}\,\boldsymbol{h}_+^\dagger \boldsymbol{h}_- -(+)\left(2\mathrm{tr}\,(\boldsymbol{h}_+^\dagger \boldsymbol{h}_-)^2 - (\mathrm{tr}\,\boldsymbol{h}_+^\dagger \boldsymbol{h}_-)^2\right)^{1/2}\right]$$

Massless spectrum if $\boldsymbol{h} = \pm\boldsymbol{\eta}$. Instead, if $\det \boldsymbol{h}_+^\dagger \boldsymbol{h}_- = 0$, we have $M_1^2 = 0$ and

$$M_2^2 = \frac{v^2}{2}\left[\mathrm{tr}\boldsymbol{h}^\dagger \boldsymbol{h} - \mathrm{tr}\boldsymbol{\eta}^\dagger \boldsymbol{\eta} - 2i\mathrm{Im}\,\mathrm{tr}\,\boldsymbol{h}^\dagger \boldsymbol{\eta}\right]$$

How many complex phases for $N = 3$? (with Madeleine Dale and Robert Mason in prep)

The Goldstone theorem

The conserved current is sufficient to ensure the **Goldstone theorem** continues to hold in the case of **global spontaneous symmetry breaking** in non-Hermitian theories (Alexandre, Ellis, PM & Seynaeve '18):

$$\mathcal{L} = \partial_\alpha \phi_1^* \partial^\alpha \phi_1 + \partial_\alpha \phi_2^* \partial^\alpha \phi_2 + m_1^2 |\phi_1|^2 - m_2^2 |\phi_2|^2 - \mu^2 (\phi_1^* \phi_2 - \phi_2^* \phi_1) - \frac{g}{4} |\phi_1|^4$$

$$\left.\begin{aligned} \frac{\partial U}{\partial \phi_1^*} &= \frac{g}{2} |\phi_1|^2 \phi_1 - m_1^2 \phi_1 + \mu^2 \phi_2 = 0 \\ \frac{\partial U}{\partial \phi_2^*} &= m_2^2 \phi_2 - \mu^2 \phi_1 = 0 \end{aligned}\right\} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sqrt{2 \frac{m_1^2 m_2^2 - \mu^4}{g m_2^2}} \begin{pmatrix} 1 \\ \frac{\mu^2}{m_2^2} \end{pmatrix} e^{i\theta}$$

Away from the exceptional point, the mass matrix for the fluctuations $\hat{\phi}_i$ about these vevs has a single zero eigenvalue, corresponding to the **Goldstone mode**

$$G \propto v_1 \mathrm{Im}\, \hat{\phi}_1 - v_2 \mathrm{Im}\, \hat{\phi}_2$$

(see also Fring & Taira '19)

Gauge invariance

Since $\partial_\alpha \partial_\beta F^{\alpha\beta} = 0$ identically, the consistency of the **Maxwell equation** suggests we should couple to the **conserved current** via the **minimal coupling prescription**:

$$\partial_\alpha F^{\alpha\beta} = j_-^\beta$$

with

$$j_-^\beta = ig\left(\phi_1^* D_+^\beta \phi_1 - \phi_1 \left[D_+^\beta \phi_1\right]^*\right) - ig\left(\phi_2^* D_-^\beta \phi_2 - \phi_2 \left[D_-^\beta \phi_2\right]^*\right)$$

$$D_\pm^\beta = \partial^\beta \pm igA^\beta$$

But the Lagrangian is then **not gauge invariant**, and we find a **longitudinal contribution** to the gauge boson **polarization tensor** at one-loop:
(see Alexandre, Ellis, PM & Seynaeve '19; PM '19)

$$p_\alpha \Pi^{\alpha\beta}(p^2 = 0) \propto g^2 \eta^2 \qquad \eta \equiv \frac{2|\mu^2|}{|m_1^2 - m_2^2|}$$

Gauge invariance (see Alexandre, Ellis, PM & Seynaeve '19)

To restore **gauge invariance**, we must couple to the **non-conserved current** j_+^ν:

$$\partial_\alpha F^{\alpha\beta} = j_+^\beta$$

with

$$j_+^\beta = ig\left(\phi_1^* D^\beta \phi_1 - \phi_1 \left[D^\beta \phi_1\right]^*\right) + ig\left(\phi_2^* D^\beta \phi_2 - \phi_2 \left[D^\beta \phi_2\right]^*\right)$$

$$D^\beta = \partial^\beta + igA^\beta$$

And to keep the field equations consistent, we must restrict to harmonic gauge functions by adding the gauge-fixing term

$$\mathcal{L} \supset -\frac{1}{2\xi}\left(\partial_\alpha A^\alpha\right)^2$$

The divergence of the Maxwell equation then yields the constraint

$$\Box \pi_0 = 2ig\mu^2(\phi_1^*\phi_2 - \phi_2^*\phi_1)$$

If the rhs is non-vanishing, it is not consistent with the Lorenz gauge condition $\partial_\alpha A^\alpha = 0$, but we find a modified **Gupta-Bleuler** condition for physical states.

The Abelian Higgs mechanism

In the case of **spontaneous symmetry breaking**, the **gauge boson** acquires a **mass**

$$M_A^2 = 2g^2\left(v_1^2 + v_2^2\right)$$

where (setting the phase to zero)

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \sqrt{2\frac{m_1^2 m_2^2 - \mu^4}{g m_2^2}} \begin{pmatrix} 1 \\ \frac{\mu^2}{m_2^2} \end{pmatrix}$$

such that the **Higgs mechanism** is also borne out.
(see Alexandre, Ellis, PM & Seynaeve '19)

The **Goldstone mode** is ($v_1^2 > v_2^2$)

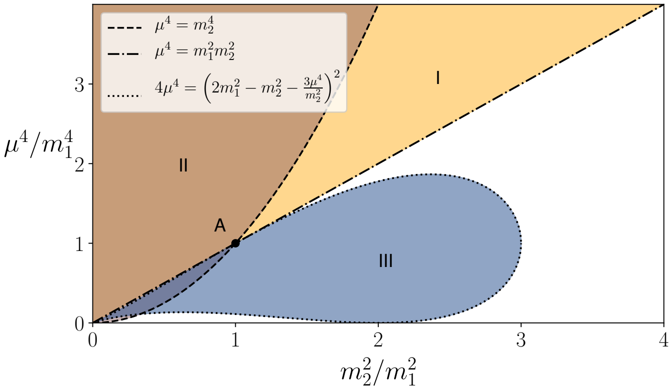
$$G = \frac{1}{\sqrt{v_1^2 - v_2^2}}\left(v_1 \mathrm{Im}\,\hat{\phi}_1 - v_2 \mathrm{Im}\,\hat{\phi}_2\right)$$

At the **exceptional point** $|\mu^2| = |m_2^2|$, $v_1^2 = v_2^2$ and the $\mathcal{PT}$ norm of the Goldstone mode diverges. Notice that $M_A^2 = 4g^2 v_1^2$ remains non-vanishing. (cf. Mannheim '19)

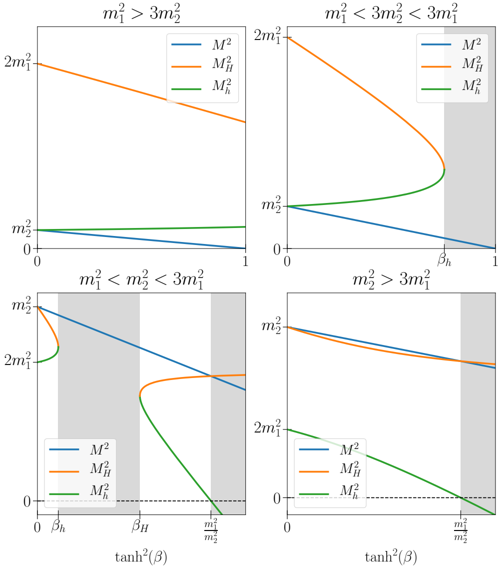
How do we make sense of the divergence of the $\mathcal{PT}$ norm here though?
$\rightarrow$ Remember that we have an **Hermitian theory on-shell** at this point!

The non-Abelian Higgs mechanism

We can promote the two complex scalar fields of the previous model to two complex $SU(2)$ doublets and realise a non-Hermitian 2HDM. (Alexandre, Ellis, PM & Seynaeve '19; see also Fring & Taira '19.)



I: symmetric phase
II: broken $\mathcal{PT}$ symmetry (masses of $H^{\pm}$, D become complex)
III: broken $\mathcal{PT}$ symmetry (masses of h and H become complex)



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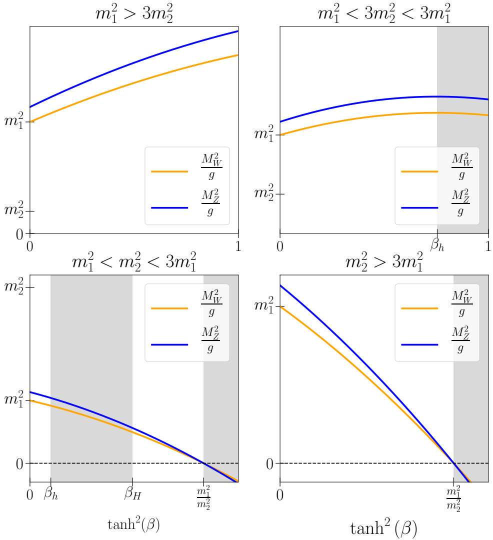

(tanh β not tan β)

Concluding remarks

- We can consistently quantize **certain non-Hermitian theories**, but there is still lots to be understood.
- The **variational procedure** for non-Hermitian QFTs has to be treated with care.
- This impacts the interpretation of **Noether's theorem**, but there still exist **conserved currents**.
- **Gauge invariance** implies minimal coupling to **non-conserved currents**.
- And the **Goldstone theorem** and **Higgs mechanism** are still borne out.
- An interesting avenue for model building **beyond the SM**?

Thank you for your attention.