A path to scale invariance in Quantum Field Theory

M. Herrero-Valea

SISSA & IFPU

The tale

- Scale Invariance
- Anomalous equivalence of frames in QFT
- Scale Invariant Quantum Field theory

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Work pre-COVID

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Scale Invariance

- Invariance under dilatations of space-time coordinates

$$x^\mu \to \Omega^{-1} x^\mu, \quad \phi \to \Omega^{d_\phi} \phi$$

- No scales — dimensional couplings — in the action

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Scale invariance =/= conformal invariance

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Standard model at high energies

$$E \gg m_i$$

CMB power spectrum

$$n_s \sim 1$$

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What if our fundamental theories are SI?

Can SI emerge dynamically?

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Provide a mechanism to stabilise the vev of σ

$$V = \lambda\left(\phi^2 - \alpha\sigma^2\right)^2, \quad \langle\phi^2\rangle = \alpha\langle\sigma^2\rangle$$

Higgs-Dilaton models. Shaposhnikov et al.

What about QFT?

There are fundamental problems to extend this idea onto QFT

Loop computations introduce a renormalisation scale μ (or cut-off scale)
This is not a scheme artefact

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For instance, Coleman-Weinberg potential for $\lambda\phi^4$

$$V^{(1)} = \frac{\lambda^2}{144\pi^2} \log\left(\frac{\phi^2}{\mu^2}\right)\phi^4$$

Scale Anomalies

While $\frac{\delta S}{\delta \omega} = 0, \quad \Omega = 1 + \omega + \ldots$

$$\frac{\delta \Gamma}{\delta \omega} \neq 0$$

For CW potential $\frac{\delta V^{(1)}}{\delta \omega} = -\frac{\lambda^2 \phi^4}{72\pi^2} \log\left(\mu^2\right)$

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This can be solved by promoting μ to a coupling through scale invariant renormalization

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Englert, Truffin, Gastmans, Wetterich, Shaposhnikov, Zenhausern, Gorbunov, Tokareva, Armillis, Monin, Ferreira, Hill, Noller, Ross, Ghilencea, Olszewski…

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A review: C. Wetterich 1901.04741

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This theory now is non-polynomial in $\boldsymbol{\sigma}$ —> EFT with cut-off $\boldsymbol{\mu}$

$$\phi^4 \log\left(\frac{\phi^2}{\lambda_\mu^2 \sigma^2}\right) = \phi^4 \log\left(\frac{\phi^2}{\mu^2}\right) - \frac{2\lambda_\mu}{\mu}\phi^2\sigma + \frac{\lambda_\mu^2}{\mu^2}\phi^4\sigma^2 + \dots$$

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This can only be done loop by loop

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Theories with scales might be the EFT expansion of a UV scale invariant theory

What is this theory?

A detour to frame equivalence

A detour to frame equivalence

Let us be specific and consider a particular SI theory

$$S_J = \int d^4x\sqrt{-g}\left(-\frac{\xi^2\phi^2}{2}R + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\lambda}{4!}\phi^4\right)$$

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Einstein Frame

$$S_E = \int d^4x\sqrt{-\tilde{g}}\left(-\frac{M_P^2}{2}\tilde{R} + \frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} + \frac{\lambda}{4!}\frac{M_P^4}{\xi^2}\right)$$

$$\tilde{g}_{\mu\nu} \to \tilde{g}_{\mu\nu}$$

$$\tilde{\phi} \to \tilde{\phi} + C$$

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$$\tilde{g}_{\mu\nu} \to \tilde{g}_{\mu\nu}$$

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$$\tilde{g}_{\mu\nu} = \frac{\xi\phi^2}{M_P^2}g_{\mu\nu}, \quad \tilde{\phi} = M_P\sqrt{\frac{1}{\xi} + 12}\,\log\left(\frac{\phi}{m}\right)$$

Frame inequivalence

Jordan Frame

$$\Gamma_J = \frac{1}{\epsilon}\int d^dx\sqrt{-g}\sum \mathcal{O}_J[R_{\mu\nu\alpha\beta}, \phi]$$

$$\frac{\delta\Gamma_J}{\delta\omega} = \int d^dx\sqrt{-g}\sum \mathcal{O}_J[R_{\mu\nu\alpha\beta}, \phi] \neq 0$$

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Einstein Frame

$$\Gamma_E = \frac{1}{\epsilon}\int d^d x\sqrt{-g}\sum \mathcal{O}_E[\tilde{R}_{\mu\nu\alpha\beta}, \tilde{\phi}]$$

$$\frac{\delta\Gamma_E}{\delta C} = 0$$

Frame inequivalence

Jordan Frame

$$\langle \nabla_\mu J_J^\mu \rangle = \sum \mathcal{O}_J[R_{\mu\nu\alpha\beta}, \phi]$$

Einstein Frame

$$\langle \nabla_\mu J_E^\mu \rangle = 0$$

But classically $J_J^\mu \equiv J_E^\mu$

Frame inequivalence

Jordan Frame

$$\langle \nabla_\mu J^\mu_J \rangle = \sum \mathcal{O}_J[R_{\mu\nu\alpha\beta}, \phi]$$

Einstein Frame

$$\langle \nabla_\mu J^\mu_E \rangle = 0$$

There is an operator difference which translates onto a physical effect

So what is going on here?

Frame inequivalence

$$\mathcal{Z}[J] = \int [d\psi] e^{-S[\psi] - \int d^4x \; J \cdot \psi}$$

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Invariant

Frame inequivalence

$$\mathscr{Z}[J] = \int [d\psi] e^{-S[\psi] - \int d^4x\, J\cdot\psi}$$

So the problem must be here

Frame inequivalence

$$\delta\mathcal{Z}[J] = \int (\delta[d\psi])\ e^{-S[\psi] - \int d^4x\ J\cdot\psi}$$

$$[d\psi] = \prod \frac{d\psi(x)}{2\pi}\sqrt{\det C}$$

$$\{\psi_1, \psi_2\} = \int d^4x \int d^4y\ \psi_1(x) C(x,y) \psi_2(y)$$

Frame inequivalence

$$\delta \mathscr{Z}[J] = \int (\delta[d\psi])\; e^{-S[\psi] - \int d^4x\; J\cdot\psi}$$

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Frame in equivalence

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If $s(x) = \Lambda^2$ we get cut-off regularisation

If $s(x) = \tilde{D}$ we find Pauli-Vilars

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If $s(x) = \tilde{D}$ we find Pauli-Vilars

Thus, the role of the metric $C(x, y)$ is that of a UV regulator (when it is field independent)

The conundrum

When we quantise a QFT in a given scheme (say cut-off), we assume

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But $C(x,y)$ is a metric in field space!! $\quad \delta\tilde{\psi} \times \tilde{C} \times \delta\tilde{\psi} = \delta\psi \times C \times \delta\psi$

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If the metric in the E frame is $C_E(x, y) = \Lambda^2 \delta(x, y)$, then $C_J = e^{2\alpha} C_E$

Where $e^{2\alpha} = \dfrac{\xi \phi^2}{M_P^2}$

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$$\mathcal{Z}[J] = \int [d\psi] e^{-S[\psi] - \int d^4x \, J \cdot \psi}$$

There is an extra determinant here $\sqrt{\det C}$

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Jordan Frame

$$\tilde{C}_J = \Lambda^2 \delta(x, y) e^{2\alpha}$$

$$\delta\tilde{\Gamma}_J = \delta\Gamma_J - \delta\mathscr{A} = 0!!$$

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Frame discriminant

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If we look for cancelling the scale anomaly, we arrive at scale invariant renormalisation!

Frame discriminant

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$$\mu \to \sqrt{\frac{\xi}{M_P^2}}\ \mu\phi = \lambda_\mu \phi\,!$$

Note, however, that this procedure is general for any field redefinition, not only for scale transformations.

Frame discriminant

$$\mathscr{Z}[J] = \int [d\psi] e^{-S[\psi] - \int d^4x \ J \cdot \psi}$$

The definition of a QFT has two ingredients

- The action
- The integration measure = the physical frame

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The definition of a QFT has two ingredients

- The action
- The integration measure = the physical frame

Once the physical degrees of freedom are identified, that fixes the physical frame

Scale Invariant Quantum Field Theory

SI QFT

We propose a SI generalisation of the path integral

$$\mathcal{S}[J] = \int (d\phi)\ e^{-S-J\cdot\phi}$$

$$(d\phi) = \prod \frac{d\phi(x)}{\sqrt{2\pi}} \sqrt{\det(\sigma^2 \times \delta(x,y))}$$

Where $\sigma(\phi)$ is an operator with scaling dimension $d_s = 1$

$$\frac{\delta \mathcal{S}}{\delta \omega} = 0$$

Perturbation theory

Let us assume that $\boldsymbol{\sigma}$ is an external dilaton field

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The problem is that the metric determinant is defined through the full theory

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$$\sqrt{\det(\Lambda^2 \times \delta(x,y))} \times \sqrt{\det(\frac{\sigma^2}{\Lambda^2} \times \delta(x,y))} = e^{\mathcal{M}} \sqrt{\det(\Lambda^2 \times \delta(x,y))}$$

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$$\mathcal{G} = \Gamma - \mathcal{M}$$

Perturbation theory

$$\mathscr{M} = \frac{1}{2}\mathrm{Tr}\log\left(\frac{\sigma^2}{\Lambda^2}\delta^{(4)}(x,y)\right) = \frac{1}{2}\int d^4x\log(\sigma^2/\Lambda^2)\times\langle\phi\,|\,\phi\rangle$$

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$$\langle\phi\,|\,\phi\rangle = \mathscr{M}_0 + \lambda\mathscr{M}_1 + \lambda^2\mathscr{M}_2 + \dots$$

$$\langle\phi\,|\,\phi\rangle = \lim_{M\to\infty}\int \frac{d^4k}{2\pi^4}\, e^{-\frac{E}{M^2}}$$

Perturbation theory

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$$\langle\phi\,|\,\phi\rangle = \mathscr{M}_0 + \lambda\mathscr{M}_1 + \lambda^2\mathscr{M}_2 + \dots \qquad \langle\phi\,|\,\phi\rangle = \lim_{M\to\infty}\int \frac{d^4k}{2\pi^4}\, e^{-\frac{E}{M^2}}$$

1- Compute $\mathscr{M}$ at order g^N using the effective equations of motion.

2 - Compute the effective action to order N

An example

$$S = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{\lambda}{4!} \phi^4 \right)$$

$$\mathscr{V}^{(1)} = V^{(1)} - \mathscr{M}^{(1)} \Big|_{p=0}$$

$$V^{(1)} = \frac{\lambda^2}{144\pi^2} \log \left(\frac{\phi^2}{\Lambda^2} \right) \phi^4$$

$$\mathcal{M} = \frac{1}{2} \int d^4x \log\left(\sigma^2/\Lambda^2\right) \int \frac{d^4k}{(2\pi)^4} \exp\left(-\frac{k^2}{M^2} - \frac{\lambda}{6M^2}\phi^2 \right) = \frac{1}{4\pi^2} \int d^4x \log\left(\sigma^2/\Lambda^2\right) \left(M^4 - \frac{\lambda M^2}{6}\phi^2 + \frac{\lambda^2}{36}\phi^4 \right), \tag{33}$$

An example

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$$\mathscr{V}^{(1)} = V^{(1)} - \mathscr{M}^{(1)} \Big|_{p=0}$$

$$\mathscr{V}^{(1)} = \frac{\lambda^2}{144\pi^2} \log\left(\frac{\phi^2}{\sigma^2} \right) \phi^4$$

This reproduces the scale invariant renormalisation procedure

Conclusions

The integration measure of the path integral is **important**
when the metric is in the game

Non-linear field redefinitions map **different** quantum field theories

We can exploit this idea to define a SI QFT

- It would be interesting to understand better this proposal
- What about a non-perturbative computation?