Scale invariant renormalization and applications

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Based on: arXiv:1712.06024 (PRD); 1612.09120 (PRD) with Z. Lalak, P. Olszewski (Warsaw)

Related: arXiv:1904.06596, 1812.08613, 1809.09174 with Hyun Min Lee (KIAS/Seoul U)

- “New physics” beyond SM: new symmetry?

- SUSY @ TeV: hierarchy problem [and other problems] solved [in theory....]

- scale invariance (SI); - SM with $m_h = 0$ has classical scale invariance. [Bardeen 1995]

$$x \to \rho\, x, \quad \phi \to \rho^{-1}\,\phi, \quad [\phi] = 1, \quad \text{SI forbids} \quad \int d^4x\; m^2\phi^2 + \cdots$$

- no dimensionful couplings; $\Rightarrow$ All scales from vev's! (including M_{Planck}!)

- classical SI: models: Higgs portal $\lambda\phi^2\,\sigma^2$, inflation... [Shaposhnikov et al; Lindner et al]

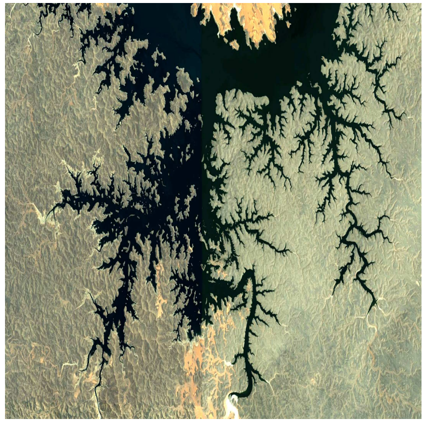
- quantum level?

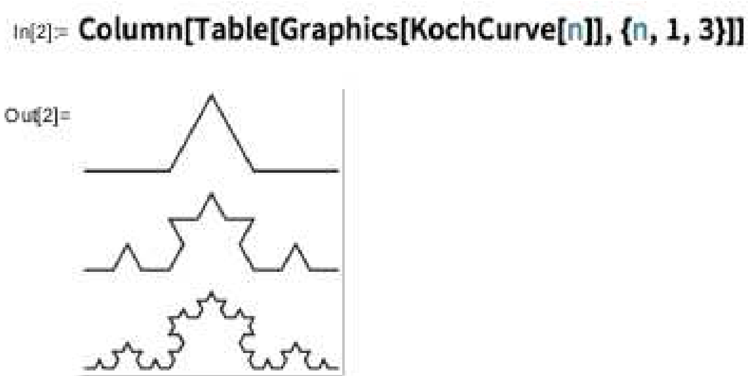
- In this talk: - flat space: SI at quantum level; protects a classical hierarchy of vev's? (spontan. ~~SI~~)

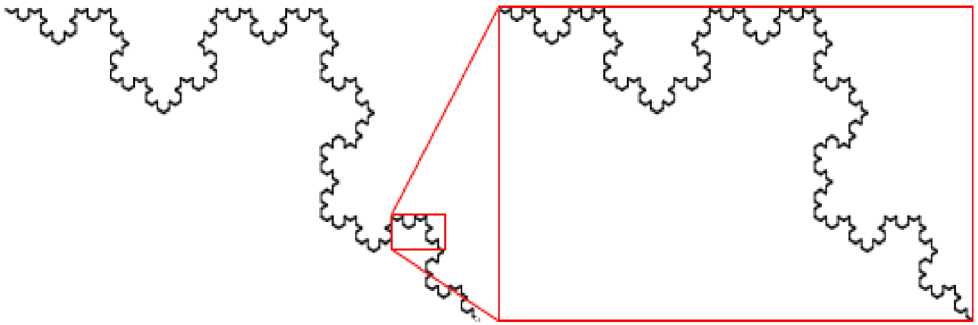
- curved space? fate of dilaton? Weyl gauge symmetry/gravity $\to$ Einstein-Proca action+cc

[Weyl conformal geometry]

- in Nature: discrete scale invariance: self-similarity across different scales [fractals]



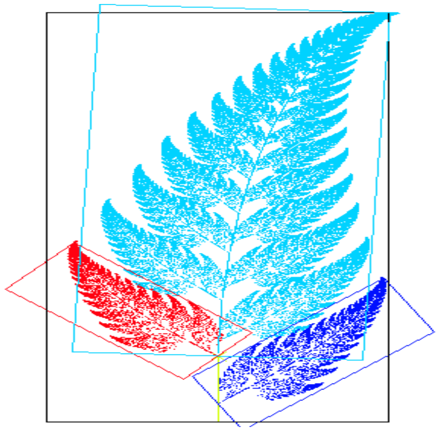


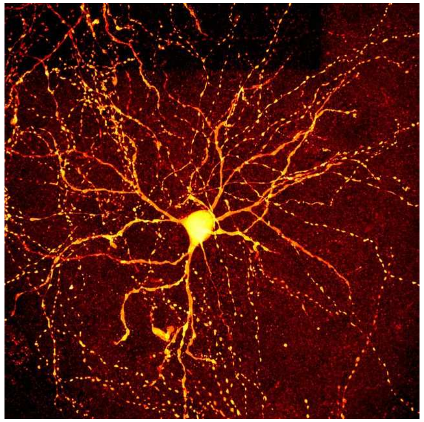


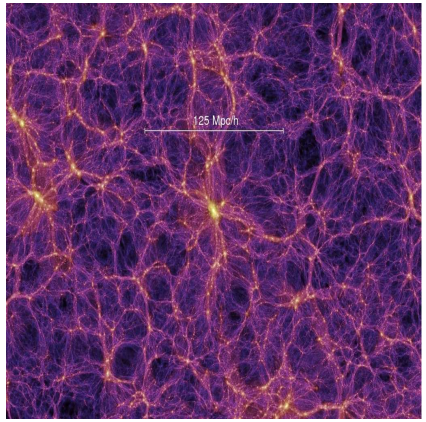
- self-repeating patterns at all length scales (new structure revealed).

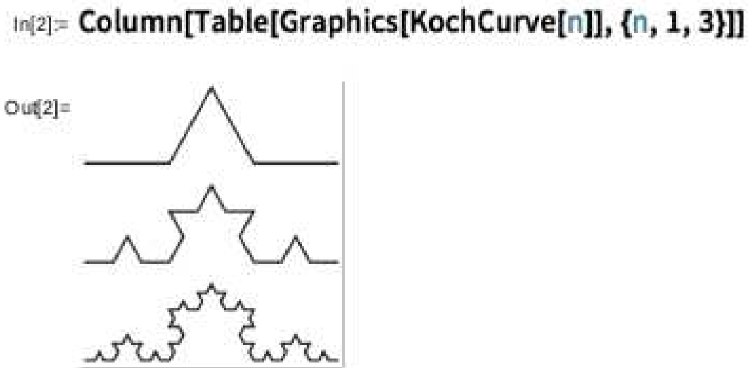
Koch curve/snowflake

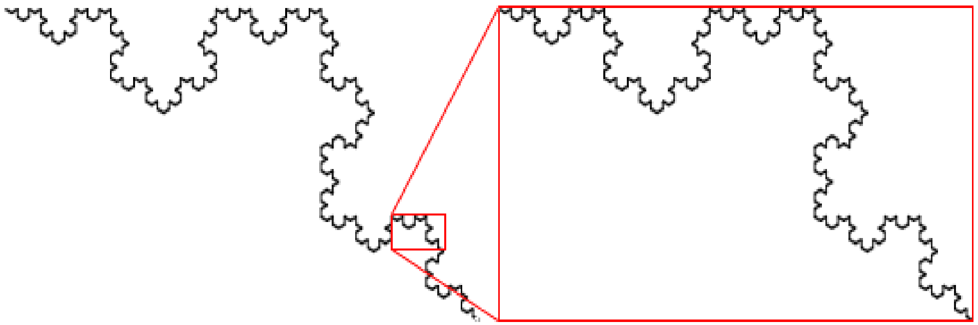
- in Nature: discrete scale invariance: self-similarity across different scales [fractals]





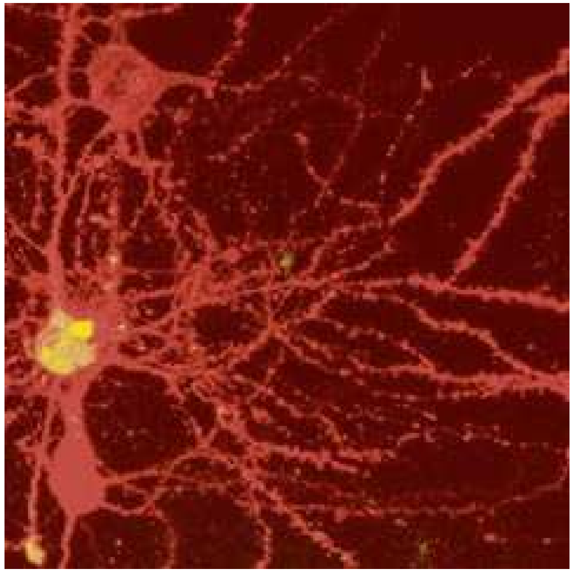


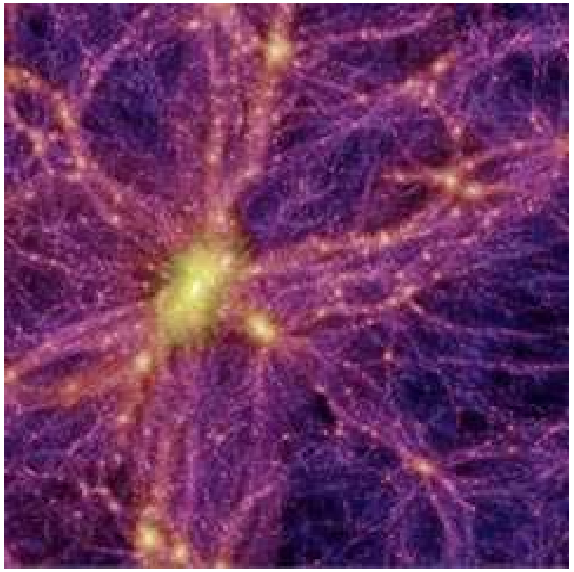




- self-repeating patterns at all length scales (new structure revealed).   Koch curve/snowflake

... self-similarity at very different scales:





Left: actual neurons synapses in the mouse brain, μm size. credit: Mark Miller (Brandeis U 2006).
Right: cluster of galaxies, stars, DM. astro-ph/0504097 credit: Virgo Consortium.
https://wwwmpa.mpa-garching.mpg.de/galform/millennium/

• "Scale Invariance" of the action: one can have:

1. - (Global) scale invariance, flat space:

$$x'_\mu = \rho\, x_\mu; \qquad \phi'(\rho\, x) = \frac{1}{\rho}\,\phi(x) \qquad \Longleftarrow\!=\!= \quad \text{in this talk.}$$

2. - Local scale invariance: (ghosts - unitarity?) [t'Hooft 1104.4543; 1410.6675, IJMP 2016]

$$g'_{\mu\nu}(x) = \Omega(x)^2 g_{\mu\nu}(x), \quad \phi'(x) = \frac{1}{\Omega(x)}\phi(x)$$

3. - Gauged local scale invariance (Weyl gauge symmetry) D.G.arXiv:1904.06596, 1812.08613

$$g'_{\mu\nu}(x) = \Omega(x)^2 g_{\mu\nu}(x), \quad \phi'(x) = \frac{1}{\Omega(x)}\phi(x), \quad \omega'_\mu = \omega_\mu - \partial_\mu \ln\Omega(x)$$

This is Weyl conformal geometry giving Weyl gravity (1918). Non-metric theory $\nabla_\mu\, g_{\alpha\beta} = \omega_\mu\, g_{\alpha\beta}$.
Einstein (1918) criticised it (due to massless ω_μ). Avoids this if ω_μ massive and decouples ($\omega_\mu\!=\!0$) at high scale

• The problem at quantum level:

- UV regularisation $\rightarrow$ dimensionful parameter $\rightarrow$ UV regulators break SI explicitly:

DR, d=4-2ϵ: $\lambda_\phi^0 = \mu^{2\epsilon}\Big[\lambda_\phi + \sum_n a_n/\epsilon^n\Big], \quad L=(1/2)(\partial_\mu\phi)^2 - \lambda_\phi\,\mu^{2\epsilon}\,\phi^4,$

[“Higgs portal” models: any flat direction lifted by loop effects. Pseudo-Goldstone (light)].

• We avoid explicit $\not{S}$I: [Englert et al 1976, I-Z book, Shaposhnikov 2008]

$\Rightarrow$ replace $\mu \rightarrow$ field σ, spontaneous $\langle\sigma\rangle\neq 0 \Rightarrow$ Goldstone (dilaton): $\sigma = \langle\sigma\rangle\, e^{\tau}$, $x\rightarrow \rho\, x;\ \sigma\rightarrow\rho^{-1}\sigma;$

$\Rightarrow$ spectrum extended by σ! different model! $\Rightarrow \tau\rightarrow\tau-\ln\rho$ shift symmetry

[M_{string} moduli dep]

• SI loop corrections? setup: SI broken at $\langle\sigma\rangle\gg\langle\phi\rangle\sim m_z$. Quantum stable? $\langle\sigma\rangle$=scale of 'new physics'

How is higgs mass protected from large quantum corrections associated with $\langle\sigma\rangle$.

• Scale invariant regularisation (SIR)

• Action: $d\!=\!4,$ $$S=\int d^4x\,\Big[(\partial_\mu\phi)^2-\underbrace{V(\phi)}_{\text{visible}}\Big]+\int d^4y\,\underbrace{L_h(\sigma,\partial\sigma)}_{\text{hidden}}$$

- spectrum extended by σ! potential: $\lambda_\sigma\,\sigma^4$ but Poincaré symmetry demands $\lambda_\sigma=0$ [Fubini 1976]
- each sector SI (shift symmetry) $\rightarrow$ enhanced shift symmetry: $S_h\times S_v\Rightarrow\lambda_m\phi^2\sigma^2$: λ_m naturally small

[Volkas, Kobakhidze, Foot 2013]

$d\!=\!4\!-\!2\epsilon:\quad \mu=z\,\sigma^{2/(d-2)},\quad V\rightarrow\tilde V=\Big[z\,\sigma^{2/(d-2)}\Big]^{4-d}V(\phi),\qquad (z:\text{dim-less}),\ \ \sigma=\langle\sigma\rangle+\tilde\sigma$

$$\tilde V=\big(z\langle\sigma\rangle^{1/(1-\epsilon)}\big)^{2\epsilon}\Big[1+2\,\epsilon\,\Big(\frac{\tilde\sigma}{\langle\sigma\rangle}-\frac{\tilde\sigma^2}{2\langle\sigma\rangle^2}+...\Big)+\epsilon^2\,\Big(\frac{2\tilde\sigma}{\langle\sigma\rangle}+....\Big)+\mathcal{O}(\epsilon^3)\Big]\,V(\phi),$$

$\Rightarrow$ SIR=DR+dilaton with ∞-many "evanescent" ϵ-couplings $\Rightarrow\epsilon^k\!\times\!(1/\epsilon^n)\Rightarrow$ new corrections/poles.

$\Rightarrow$ expect c-terms: $\dfrac{\partial^{2n}\alpha^{m+4}}{\sigma^{2n+m}},\quad m,n\geq 0\quad$ [$\sigma\sigma\rightarrow\sigma\sigma$ at 3 loops $(\partial_\mu\ln\sigma)^4$]; $\qquad\langle\sigma\rangle=$ 'new physics'.

• One-loop SI potential $(d = 4 - 2\epsilon)$:

$$L = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - V(\phi), \qquad V(\phi) = \frac{\lambda}{4!}\,\phi^4 \to \tilde{V}(\phi, \sigma) = z^{2\epsilon}\,\sigma^{2\epsilon/(1-\epsilon)}\,V(\phi).$$

$$V_1 \;=\; \tilde{V} - \frac{i}{2}\int \frac{d^d p}{(2\pi)^d}\ \mathrm{tr}\,\ln\Big[p^2 - \tilde{V}_{\alpha\beta} + i\varepsilon\Big] = \tilde{V} + \frac{1}{4\kappa}\sum_{s=\phi,\sigma} \tilde{M}_s^4\,\Big[\frac{-1}{\epsilon} + \ln\frac{\tilde{M}_s^2}{c_0}\Big].$$

$\tilde{M}_\phi^4 = M_\phi^4 + \epsilon ...,\ \tilde{M}_\sigma^4 \sim \epsilon^2$. Then

$$\delta L_1 \;=\; -\mu(\sigma)^{2\epsilon}\,\frac{1}{4!}\,(Z_\lambda - 1)\lambda\,\phi^4 \quad \text{with} \quad Z_\lambda - 1 = \frac{3\,\lambda}{2\kappa\epsilon}, \qquad \kappa = (4\pi)^2$$

$$U \;=\; V(\phi) + \frac{1}{4\kappa}V_{\phi\phi}^2\,\Big[\overline{\ln}\frac{V_{\phi\phi}}{(z\,\sigma)^2} - \frac{1}{2}\Big]; \qquad V_{\phi\phi} = \frac{\lambda}{2}\,\phi^2.$$

$\Rightarrow$ Scale invariant result, due to dilaton σ. ($\ln\sigma$: finite quantum effect, due to symmetry, not dynamics).

$\overline{\ln} A = \ln A/(4\pi e^{1-\gamma})$

- One-loop SI potential:

No new poles: one-loop beta $\beta_\lambda^{(1)}$ unchanged from the theory without dilaton: $\lambda^B = \lambda Z_\lambda\, Z_\phi^{-2}$.

$$\frac{d\,\lambda^B}{d\ln z} = 0 \quad \Rightarrow \quad \beta_\lambda^{(1)} = \frac{d\lambda}{d\ln z} = \frac{3}{\kappa}\,\lambda^2,$$

[Shaposhnikov et al, Tamarit]

Callan-Symanzik:

$$\frac{dU}{d\ln z} = \Big(\frac{\partial}{\partial \ln z} + \beta_\lambda^{(1)}\frac{\partial}{\partial\lambda}\Big)\, U = O(\lambda^3).$$

Decouple dilaton fluctuations

$$U \;=\; V(\phi) + \underbrace{\frac{1}{4\kappa}\,V_{\phi\phi}^2\Big[\overline{\ln}\frac{V_{\phi\phi}}{(z\langle\sigma\rangle)^2} - \frac{1}{2}\Big]}_{CW} + \underbrace{\frac{1}{4\kappa}\,V_{\phi\phi}^2\,\Big(-\frac{\tilde\sigma}{\langle\sigma\rangle} + \frac{1}{2}\frac{\tilde\sigma^2}{\langle\sigma\rangle^2} + \cdots\Big)}_{\to 0;\ \text{small yet maintains SI}}, \qquad V_{\phi\phi} \equiv \frac{\lambda}{2}\,\phi^2.$$

$\sigma = \langle\sigma\rangle + \tilde\sigma$. CW term also respects CS wrt $\mu_0 = z\langle\sigma\rangle$.

$\Rightarrow$ in theories with explicit $\not{S}$I by quantum calculations (no dilaton) $\beta_\lambda = 0$ is a necessary condition for SI

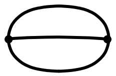
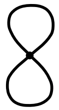
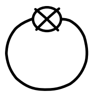
$\Rightarrow$ here $\beta_\lambda = 0$ is not a necesary condition for SI; spontaneous $\not{S}$I, different spectrum/sym! $\partial_\mu D^\mu = 0$

[C. Tamarit 2014]

• Two-loop SI potential: using: $\tilde V(\phi,\sigma) = z^{2\epsilon}\,\sigma^{2\epsilon/(1-\epsilon)}\,V(\phi) \sim \sigma^{2\epsilon}\,V(\phi)$ [background field method]

$$\Rightarrow \tilde V(\phi+\delta_\phi,\sigma+\delta_\sigma) = \tilde V(\phi,\sigma)+\tilde V_\alpha\delta_\alpha+\frac{1}{2}\,\tilde V_{\alpha\beta}\,\delta_\alpha\delta_\beta+\frac{1}{3!}\,\tilde V_{\alpha\beta\gamma}\,\delta_\alpha\delta_\beta\delta_\gamma+\frac{1}{4!}\,\tilde V_{\alpha\beta\gamma\rho}\,\delta_\alpha\delta_\beta\delta_\gamma\delta_\rho+\cdots \quad \alpha,\beta=\phi,\sigma.$$

$$\tilde V_{\alpha\beta\ldots} = \partial_\alpha\partial_\beta..\tilde V$$

$V_2 = \frac{i}{12}$  $+\frac{i}{8}$  $+\frac{i}{2}$  $= \frac{i}{12}\,\tilde V_{\alpha\beta\gamma}\tilde V_{\alpha'\beta'\gamma'}\int\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\,(\tilde D_p)_{\alpha\alpha'}\,(\tilde D_q)_{\beta\beta'}\,(\tilde D_{p+q})_{\gamma\gamma'}+\cdots$

$$= (z\sigma)^{2\epsilon}\,\frac{\lambda^3\,\phi^4}{32\kappa^2}\Big\{-\frac{3}{\epsilon^2}+\frac{2}{\epsilon}+\mathcal{O}(\epsilon^0)\Big\}; \qquad (\tilde D_p)_{\alpha\beta} = (D_p)_{\alpha\beta}+\epsilon\,(\ldots)_{\alpha\beta}+\epsilon^2\,(\ldots)_{\alpha\beta}$$

same poles, ϵ-shifts to propagators, vertices: $\tilde V_{\alpha\beta\gamma\ldots} = V_{\alpha\beta\gamma\ldots}+\epsilon\,(\ldots)_{\alpha\beta\gamma\ldots}+\epsilon^2\,(\ldots)_{\alpha\beta\gamma\ldots}$

[Lalak, Olszewski,DG]
[1712.06024]

Two-loop corrected U

$$U = \frac{\lambda}{4!}\,\phi^4\,\Big\{1+\frac{3\lambda}{2\kappa}\Big(\overline{\ln}\frac{V_{\phi\phi}}{(z\sigma)^2}-\frac{1}{2}\Big)+\frac{3\lambda^2}{4\kappa^2}\Big(4+A_0-4\,\overline{\ln}\frac{V_{\phi\phi}}{(z\sigma)^2}+3\,\overline{\ln}^2\frac{V_{\phi\phi}}{(z\sigma)^2}\Big)+\frac{5\lambda^2}{\kappa^2}\frac{\phi^2}{\sigma^2}+\frac{7\lambda^2}{24\kappa^2}\frac{\phi^4}{\sigma^4}\Big\},$$

V $\quad V^{(1)}$ $\quad V^{(2)}$ $\quad$ new $V^{(2,n)}$ finite z-independent

- Two-loop: Taylor expand about $\sigma = \langle\sigma\rangle + \tilde{\sigma}$:

$$U = \frac{\lambda}{4!}\phi^4 \Big\{ 1 + \frac{3\lambda}{2\kappa}\Big(\overline{\ln}\frac{V_{\phi\phi}}{(z\langle\sigma\rangle)^2} - \frac{1}{2}\Big) + \frac{3\lambda^2}{4\kappa^2}\Big(4 + A_0 - 4\,\overline{\ln}\frac{V_{\phi\phi}}{(z\langle\sigma\rangle)^2} + 3\,\overline{\ln}^2\frac{V_{\phi\phi}}{(z\langle\sigma\rangle)^2}\Big) + \mathcal{O}\Big(\frac{1}{\langle\sigma\rangle}\Big)$$

- This is the "usual" CW result with $\mu = \langle\sigma\rangle$, broken SI, no dilaton present. [Cheng, I. Jack, T. Jones, S. Martin]

- new terms comparable/larger than standard two-loop terms

$$\frac{\phi^n}{\sigma^n} \sim 1., \qquad n = 1, 2; \qquad\qquad \frac{\phi}{\sigma} = \frac{\phi}{\langle\sigma\rangle}\Big(1 - \frac{\tilde{\sigma}}{\langle\sigma\rangle} + \frac{\tilde{\sigma}^2}{\langle\sigma\rangle^2} + \cdots\Big), \qquad \rightarrow 0, \text{ if } \phi \ll \sigma$$

$\Rightarrow$ Non-polynomial terms: - SI, vanish if $\phi \ll \sigma$, only $\log\sigma$ left; no $\lambda^n\phi^2\sigma^2 = \lambda^n\langle\sigma\rangle^2\phi^2 + \ldots$

$\rightarrow$ no tuning of λ ($\lambda =$ higgs self-coupling)

- finite c-terms, cannot be seen in a scheme that breaks this symmetry.

- not forbidden by symmetry $\rightarrow$ ops quantum generated; non-renormalizability!

• Two-loop: No new poles. Two-loop $\beta_\lambda^{(2)}$, anom dims $\gamma_\phi^{(2)}$ unchanged. Usual counterterm:

$$\delta L_2 = \frac{1}{2}(\partial_\mu\phi)^2\,\delta_\phi^{(2)} - \mu(\sigma)^{2\epsilon}\frac{1}{4!}\lambda\,\phi^4\,\delta_\lambda^{(2)}, \qquad \delta_\lambda^{(2)} = \frac{\lambda^2}{\kappa^2}\Big(\frac{9}{4\,\epsilon^2} - \frac{3}{2\,\epsilon}\Big), \qquad \delta_\phi^{(2)} = \frac{-\lambda^2}{24\,\kappa^2\,\epsilon}.$$

$$\beta_\lambda^{(2)} = -\frac{17}{3\,\kappa^2}\lambda^3, \qquad \text{unchanged (as if no dilaton \& } \mu = \text{const).} \qquad \gamma_\sigma^{(2)} = 0.$$

Callan-Symanzik (check):

$$\frac{\partial V^{(2)}}{\partial \ln z} + \Big[\beta_\lambda^{(2)}\frac{\partial}{\partial\lambda} + \gamma_\phi^{(2)}\phi\frac{\partial}{\partial\phi} + \gamma_\sigma^{(2)}\sigma\frac{\partial}{\partial\sigma}\Big]V + \beta_\lambda^{(1)}\frac{\partial V^{(1)}}{\partial\lambda} = O(\lambda^4), \quad \text{“usual” CS}$$

$$\frac{\partial V^{(2,\mathrm{n})}}{\partial \ln z} = O(\lambda^4), \qquad \beta^{(k)},\ \gamma^{(k)},\ V^{(k)}\text{: k-loop correction.}$$

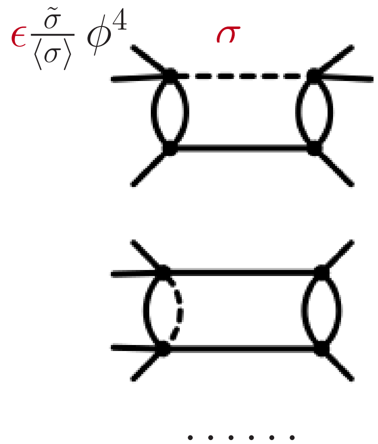
• if $V \to V + \frac{\lambda_m}{2}\phi^2\sigma^2$, $\quad V^{(2,n)} \supset \frac{\lambda\,\lambda_m}{96\kappa^2\,\epsilon}\Big\{7\,(2\lambda_m - \lambda)\frac{\phi^6}{\sigma^2} - \frac{3\lambda_m}{2}\frac{\phi^8}{\sigma^4}\Big\} \;\Rightarrow\; \beta_\lambda \to \beta_\lambda + \frac{\lambda_m}{\kappa^2}\Big[12\lambda_m^2 - 7\lambda_m\lambda - 40\lambda^2\Big]$

$\beta_{\lambda_m} \propto \lambda_m^2$; $\quad \lambda_m \to 0$: $S_v \times S_h$.

- Three-loop SI potential: Counterterms! [Monin 2015]

$$\delta L_3 = \frac{1}{2}\,\delta_\phi^{(3)}\,(\partial_\mu\phi)^2 - \mu^{2\epsilon}\Big\{\frac{1}{4!}\,\delta_\lambda^{(3)}\,\lambda\,\phi^4 + \frac{1}{6}\,\delta_{\lambda_6}^{(3)}\,\lambda_6\,\frac{\phi^6}{\sigma^2} + \frac{1}{8}\,\delta_{\lambda_8}^{(3)}\,\lambda_8\,\frac{\phi^8}{\sigma^4}\Big\}$$

$$\delta_{\lambda_6}^{(3)} = \frac{3}{2}\frac{\lambda^4}{\lambda_6\,\kappa^3\,\epsilon}, \qquad \delta_{\lambda_8}^{(3)} = \frac{275}{864}\frac{\lambda^4}{\lambda_8\,\kappa^3\,\epsilon.} \;\Rightarrow\; \gamma_\phi^{(3)},\, \beta_{\lambda_6},\, \beta_{\lambda_8} = \ldots.$$



Integrate CS: Three-loop term: $U_3 = \Delta V + V^{(3)} + V^{(3,n)}$. $V^{(3)}$='old'$[\,\mu \to z\sigma\,]$; new: $V^{(3,n)}$ and ΔV

$$\Delta V = \frac{\lambda_6\,\phi^6}{\sigma^2} + \frac{\lambda_8\,\phi^8}{\sigma^4}, \qquad V^{(3)} = \frac{\lambda^4\,\phi^4}{\kappa^3}\Big\{\mathcal{Q} + \Big(\frac{97}{128} + \frac{9}{64}A_0 + \frac{\zeta[3]}{4}\Big)\,\overline{\ln}\frac{V_{\phi\phi}}{(z\sigma)^2} - \frac{31}{96}\overline{\ln}^2\frac{V_{\phi\phi}}{(z\sigma)^2} + \frac{9}{64}\overline{\ln}^3\frac{V_{\phi\phi}}{(z\sigma)^2}\Big\}$$

$$V^{(3,n)} = \frac{\lambda^3\,\phi^4}{2\,\kappa^3}\Big\{\Big(27\lambda - \frac{\lambda_6}{2}\Big)\frac{\phi^2}{8\,\sigma^2} + \Big(\frac{401\,\lambda}{72} - \lambda_8\Big)\frac{\phi^4}{16\,\sigma^4}\Big\}\,\overline{\ln}\frac{V_{\phi\phi}}{(z\sigma)^2}. \qquad V_{\phi\phi} \equiv \frac{\lambda}{2}\phi^2.$$

[DG 1712.06024]

$\Rightarrow$ SI effective operators always suppressed by $\sigma \Rightarrow$. At large σ enhanced symmetry $S_v \times S_h$ restored $\Rightarrow$

$\Rightarrow$ forbids c-terms $\lambda^2\,\phi^2\sigma^2 = \lambda^2\,\langle\sigma\rangle^2\,\phi^2 + \cdots$ No tunning of Higgs selfcoupling λ for large σ.

• Dilatation current D_μ: $$D^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_j)}(x^\nu \partial_\nu \phi_j + d_\phi) - x^\mu \mathcal{L}, \qquad d_\phi = (d-2)/2 \quad \text{(scalars)},$$

- in $d = 4 - 2\epsilon$, potential $\tilde{V}$: $$\partial_\mu D^\mu = (d_\phi + 1)(\partial_\mu \phi_j)\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_j)} + d_\phi \, \phi_j \frac{\partial \mathcal{L}}{\partial \phi_j} - d\mathcal{L}$$
$$= d\tilde{V} - \frac{d-2}{2}\phi_j \frac{\partial \tilde{V}}{\partial \phi_j}, \quad \phi_j = \phi, \sigma; \quad \text{(onshell, canonical k.t.)}$$

• in SI theory: $\tilde{V}$ homogeneous in d dim's: $\tilde{V}(\rho\, \phi_j) = \rho^{2d/(d-2)}\, \tilde{V}(\phi_j) \Rightarrow \partial_\mu D^\mu = 0.$

• in "usual" reg: $\mu =$const, no σ: $\tilde{V} = \mu^{2\epsilon} V(\phi)$, and V is scale inv in $d = 4$:

$$\partial_\mu D^\mu = 2\,\epsilon\, \mu^{2\epsilon} V \sim 2\,\epsilon\, \mu^{2\epsilon}\Big[\lambda + \frac{\beta_\lambda}{\epsilon} + \cdots\Big]\frac{\partial V}{\partial \lambda} \propto \beta_\lambda \frac{\partial V}{\partial \lambda}, \quad [=0 \text{ only if } \beta_\lambda = 0]. \quad \text{(scale anomaly)}$$

- different field content! [Shaposhnikov et al, Tamarit 2013]

"For scale invariance, though, the situation is hopeless; any cutoff procedure necessarily involves a large mass and a large mass necessarily breaks scale invariance in a large way. This argument does not show that the occurence of anomalies is inevitable [.....]" (S. Coleman: Aspects of Symmetry, p.82, 1985).

• SM+dilaton: one-loop SI potential

[Shaposhnikov et al 2008; DG, Lalak, Olszewski]

$$V = \frac{\lambda_\phi}{3!}(H^\dagger H)^2 + \frac{\lambda_m}{2}(H^\dagger H)\,\sigma^2 + \frac{\lambda_\sigma}{4!}\sigma^4 + \cdots$$

$$= \frac{\lambda_\phi}{4!}\phi^4 + \frac{\lambda_m}{4}\phi^2\sigma^2 + \frac{\lambda_\sigma}{4!}\lambda_\sigma\,\sigma^4 + \cdots\,; \qquad (\lambda_m < 0),$$

$$\to \frac{\lambda_\phi}{4}\Big(\phi^2 + \frac{3\lambda_m}{\lambda_\phi}\sigma^2\Big)^2 + \text{loops}.$$

min $\langle\sigma\rangle \neq 0$:

1) $9\lambda_m^2 = \lambda_\phi\lambda_\sigma + \text{loops}$

2) $\dfrac{\langle\phi\rangle^2}{\langle\sigma\rangle^2} = \dfrac{-3\lambda_m}{\lambda_\phi}(1 + \text{loops})$

$\not{S}$I : tuning (i) or $V_{\text{min}} = 0 \Rightarrow$ EWSB; $m_{\tilde{\phi}} \approx \lambda_\phi\langle\phi\rangle^2 = (-3)\lambda_m\langle\sigma\rangle^2 \ll \langle\sigma\rangle^2$ if $|\lambda_m| \ll \lambda_\phi$ one classical tuning

- SM+dilaton: one-loop SI potential

[Shaposhnikov et al 2008; DG, Lalak, Olszewski]

$$V = \frac{\lambda_\phi}{3!}(H^\dagger H)^2 + \frac{\lambda_m}{2}(H^\dagger H)\,\sigma^2 + \frac{\lambda_\sigma}{4!}\sigma^4 \cdots$$
$$= \frac{\lambda_\phi}{4!}\phi^4 + \frac{\lambda_m}{4}\phi^2\sigma^2 + \frac{\lambda_\sigma}{4!}\lambda_\sigma\sigma^4 + \ldots$$
$$\to \frac{\lambda_\phi}{4}\Big(\phi^2 + \frac{3\lambda_m}{\lambda_\phi}\sigma^2\Big)^2 + \text{loops}.$$

min $\langle\sigma\rangle \neq 0$:

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$$V^{(1)} \equiv \sum_{j=\phi,\sigma;G,t,W,Z} \frac{n_j\, m_j^4(\phi,\sigma)}{4\,\kappa}\,\ln\frac{m_j^2(\phi,\sigma)}{c_j\,(z\sigma)^2}, \qquad \text{'old CW' } [\mu \to z\,\sigma]$$

$$V^{(1,n)} \equiv \frac{1}{48\kappa}\Big[(-16\lambda_m\lambda_\phi - 18\lambda_m^2 + \lambda_\phi\lambda_\sigma)\,\phi^4 - \lambda_m(48\lambda_m + 25\lambda_\sigma)\phi^2\sigma^2 - 7\lambda_\sigma^2\sigma^4$$
$$+ \lambda_\phi\lambda_m\frac{\phi^6}{\sigma^2}\Big],$$

large σ : $S_v \times S_h$

$\to 0$ if $\phi \ll \sigma$ or $\lambda_m \to 0.$

no $\lambda_\phi^2\phi^2\sigma^2 \sim \lambda_\phi\,\phi^2\langle\sigma\rangle^2$

- SM+dilaton: one-loop SI potential [Shaposhnikov et al 2008; DG, Lalak, Olszewski]

$$V = \frac{\lambda_\phi}{3!}(H^\dagger H)^2 + \frac{\lambda_m}{2}(H^\dagger H)\,\sigma^2 + \frac{\lambda_\sigma}{4!}\sigma^4 + \frac{4\,\lambda_6}{3}\frac{(H^\dagger H)^3}{\sigma^2} + \cdots$$

$$= \frac{\lambda_\phi}{4!}\phi^4 + \frac{\lambda_m}{4}\phi^2\sigma^2 + \frac{\lambda_\sigma}{4!}\lambda_\sigma\,\sigma^4 + \frac{\lambda_6}{6}\frac{\phi^6}{\sigma^2}\cdots;\ (\lambda_m<0),$$

$$\rightarrow \frac{\lambda_\phi}{4}\Big(\phi^2 + \frac{3\lambda_m}{\lambda_\phi}\sigma^2\Big)^2 + \text{loops}.$$

min $\langle\sigma\rangle\neq 0$:

1) $9\lambda_m^2 = \lambda_\phi\lambda_\sigma + \text{loops}$

2) $\frac{\langle\phi\rangle^2}{\langle\sigma\rangle^2} = \frac{-3\lambda_m}{\lambda_\phi}(1+\text{loops})$

$\not{S}\!I$: tuning (i) or $V_{\text{min}}=0 \Rightarrow$ EWSB; $m_{\tilde\phi} \approx \lambda_\phi\langle\phi\rangle^2 = (-3)\lambda_m\langle\sigma\rangle^2 \ll \langle\sigma\rangle^2$ if $|\lambda_m| \ll \lambda_\phi$ one classical tuning

$$V^{(1)} \equiv \sum_{j=\phi,\sigma;G,t,W,Z} \frac{n_j\, m_j^4(\phi,\sigma)}{4\,\kappa}\ln\frac{m_j^2(\phi,\sigma)}{c_j\,(z\sigma)^2},$$

'old CW' $[\mu \rightarrow z\,\sigma]$

$$V^{(1,n)} \equiv \frac{1}{48\kappa}\Big[(-16\lambda_m\lambda_\phi - 18\lambda_m^2 + \lambda_\phi\lambda_\sigma)\,\phi^4 - \lambda_m(48\lambda_m + 25\lambda_\sigma)\phi^2\sigma^2 - 7\lambda_\sigma^2\sigma^4$$

$$+ (\lambda_\phi\lambda_m + 6\lambda_6\lambda_\sigma)\frac{\phi^6}{\sigma^2} + 8\,(4\lambda_\phi - 2\lambda_m)\,\lambda_6\frac{\phi^8}{\sigma^4} + (192\lambda_6 + 2\lambda_\phi)\lambda_6\frac{\phi^{10}}{\sigma^6} + 40\lambda_6^2\frac{\phi^{12}}{\sigma^8}\Big],$$

large σ : $S_v \times S_h$

$$\Delta V = \frac{\lambda_p}{p}\frac{\phi^p}{\sigma^{p-4}},\ \ p = 6, 8, 10, 12.$$

$\rightarrow 0$ if $\phi \ll \sigma$ and $\lambda_m \rightarrow 0$.

no $\lambda_\phi^2\phi^2\sigma^2 \sim \lambda_\phi\,\phi^2\langle\sigma\rangle^2$

• SM+dilaton: one-loop implications recall $\lambda_\sigma \ll |\lambda_m| \ll \lambda_\phi$ (*)

minimum $\qquad \frac{\langle\phi\rangle^2}{\langle\sigma\rangle^2} = \frac{-3\lambda_m}{\lambda_\phi}[1+\zeta] \ll 1,$ if $|\lambda_m| \ll \lambda_\phi, \qquad \zeta =$ loop correction function(λ_ϕ)

• one-loop correction

$$\Delta m^2_{\tilde\phi} = \frac{-\lambda_m}{\lambda_\phi}\frac{\langle\sigma\rangle^2}{16\kappa}\Big\{27\Big[g^4\Big(\ln\frac{g^2}{4}+\frac{1}{3}\Big)+2\,g_2^4\Big(\ln\frac{g_2^2}{4}+\frac{1}{3}\Big)-16\,h_t^4\Big(\ln\frac{h_t^2}{2}-\frac{1}{3}\Big)\Big]$$
$$+\,4\,\lambda_\phi^2\Big[5\ln\frac{\lambda_\phi^2}{12}-8+\ln 27\Big]\Big\} = \lambda_m\,\lambda_\phi\langle\sigma\rangle^2+\cdots \text{ but no } \lambda_\phi^n\langle\sigma\rangle^2.$$

- no tuning of λ_m beyond classical tuning: $\beta_{\lambda_m} \propto \lambda_m$, so λ_m stays ultraweak, technically natural.
- classical hierarchy of vev's: $\langle\phi\rangle \ll \langle\sigma\rangle$ protected by quantum scale inv $S_v \times S_h$, (all orders, spont $\not{S}$I)
- recall that all scales only from fields' vev's ⇒ result is not a DR artefact! More scalars?

• SM+dilaton: one-loop beta functions (similar to when $\mu =$ constant, same field content):

$$\beta_{\lambda_\phi} = \frac{1}{\kappa}\Big[3\Big(\frac{9}{4}g_2^4+\frac{3}{4}g_1^4+\frac{3}{2}g_1^2g_2^2-12h_t^4\Big)-4\lambda_\phi\Big(\frac{3}{4}g_1^2+\frac{9}{4}g_2^2-3h_t^2\Big)+4\lambda_\phi^2+3\lambda_m^2+96\lambda_m\lambda_6\Big]$$

$$\beta_{\lambda_m} = \frac{2\lambda_m}{\kappa}\Big[\lambda_\phi+2\lambda_m+\frac{1}{2}\lambda_\sigma-\Big(\frac{3}{4}g_1^2+\frac{9}{4}g_2^2-3h_t^2\Big)\Big] \propto \lambda_m, \quad \Rightarrow \lambda_m \text{ stays small (f.p.).}$$

• for new couplings

$$\beta_{\lambda_6} = \frac{3\lambda_6}{\kappa}\Big[6\lambda_\phi-8\lambda_m+\lambda_\sigma-2\Big(\frac{3}{4}g_1^2+\frac{9}{4}g_2^2-3h_t^2\Big)\Big]$$

$$\beta_{\lambda_8} = \frac{2}{\kappa}\Big[2\lambda_6\,(28\lambda_6+\lambda_m)-4\lambda_8\Big(\frac{3}{4}g_1^2+\frac{9}{4}g_2^2-3h_t^2\Big)\Big]$$

$$\beta_{\lambda_{10}} = 10\Big[4\lambda_6^2-\lambda_{10}\Big(\frac{3}{4}g_1^2+\frac{9}{4}g_2^2-3h_t^2\Big)\Big]$$

$$\beta_{\lambda_{12}} = 2\Big[3\lambda_6^2-6\lambda_{12}\Big(\frac{3}{4}g_1^2+\frac{9}{4}g_2^2-3h_t^2\Big)\Big]$$

$\Rightarrow$ if one sets $\lambda_{6,8,10,..}=0$ at tree level, then $\beta_{\lambda_{6,8,10,12}}=0$ at one-loop, but emerge at 2-loops.

• Curved spacetime - Gravity breaks $S_v \times S_h$ by $(\xi_1\phi^2+\xi_2\sigma^2)R \Rightarrow \beta_{\lambda_m} = \xi_1(\ldots)+\xi_2(\ldots)+\lambda_m(\ldots)$

- what happens to the dilaton/flat direction? hierarchy of vev's (Higgs vs dilaton)?

• The fate of the dilaton: $\Rightarrow$ Local scale symmetry: inv. under: $\hat{g}_{\mu\nu} = \Omega^2 \, g_{\mu\nu}, \quad \hat{\sigma} = \frac{\sigma}{\Omega} \quad (a)$

wanted: $L_{EH} = -\frac{1}{2}\sqrt{g} M_p^2 \; R \; \Rightarrow \; L_1 = -\sqrt{g}\,\frac{1}{2}\left[\frac{1}{6}\,\sigma^2\,R + g^{\mu\nu}\,\partial_\mu\sigma\,\partial_\nu\sigma\right]$ (inv.)

1) ghost! 2) Fake conf symmetry?[Jackiw, Pi 2015], 3) Go to Einstein frame: # dof violated! To avoid!!

• Weyl gravity/gauge symmetry: $\hat{g}_{\mu\nu} = \Omega^2 \, g_{\mu\nu}, \quad \hat{\sigma} = \frac{\sigma}{\Omega}, \quad \hat{\omega}_\mu = \omega_\mu - \frac{1}{q}\,\partial_\mu \ln \Omega^2 \quad (b)$

$\tilde{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + \frac{q}{2}\left[\delta^\rho_\mu\,\omega_\nu + \delta^\rho_\nu\,\omega_\mu - g_{\mu\nu}\,\omega^\rho\right]$, inv (b) $\Rightarrow$ $\tilde{R} = R - 3\,q\,D_\mu\omega^\mu - \frac{3}{2}q^2\,\omega^\mu\omega_\mu. \;\Rightarrow\; \hat{\tilde{R}} = \frac{\tilde{R}}{\Omega^2}$

$\Gamma^\rho_{\mu\nu}$: Levi-Civita. $\tilde{\Gamma}^\rho_{\mu\nu}$: Weyl geom $\tilde{\nabla}_\mu\, g_{\alpha\beta} = -q\,\omega_\mu\, g_{\alpha\beta};$ $\quad \tilde{D}_\mu\sigma = (\partial_\mu - q/2\,\omega_\mu)\,\sigma \;\Rightarrow\; \hat{\tilde{D}}_\mu\hat{\sigma} = \frac{\tilde{D}_\mu\sigma}{\Omega}$

$\Rightarrow$ if $\omega_\mu \to 0$: $\tilde{\Gamma}^\rho_{\mu\nu} \to \Gamma^\rho_{\mu\nu}$, Weyl geometry $\to$ Riemann geometry; $\tilde{R} \to R$, Weyl tensor $\tilde{C}_{\mu\nu\rho\sigma} \to C_{\mu\nu\rho\sigma}$

$\Rightarrow$ invariants of (b): $\sqrt{g}\,\tilde{R}^2, \;\; \sqrt{g}\,\sigma^2\tilde{R}, \;\; \sqrt{g}\,(\tilde{D}_\mu\sigma)^2, \;\; \sqrt{g}\,F^2_{\mu\nu}, \;\; \sqrt{g}\,\tilde{C}^2_{\mu\nu\alpha\beta}$ where $F_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu$

$\tilde{X}$ denotes a quantity in Weyl geometry

• The fate of the dilaton: Weyl gravity=Einstein gravity + massive ω_μ [DG 1812.08613, 1904.06596]

$$L_2 = \sqrt{g}\,\Big\{\frac{\xi_0}{4!}\,\tilde R^2 - \frac{1}{4}\,F_{\mu\nu}^2\Big\}, \quad \xi_0>0, \qquad \text{original Weyl action (1918)}$$

$$= \sqrt{g}\,\Big\{\frac{\xi_0}{4!}\big(-2\,\sigma^2\,\tilde R - \sigma^4\big) - \frac{1}{4}\,F_{\mu\nu}^2\Big\} \qquad \text{eom: } \sigma^2 = -\tilde R; \quad \text{dilaton: } \ln\sigma \to \ln\sigma - \ln\Omega$$

+ Gauss-Bonnet + Weyl-tensor-squared.

• Weyl gauge transformation (b) - of "gauge fixing!"

$$\Omega = \frac{\xi_0\,\sigma^2}{6M_p^2} \Rightarrow \hat g_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \hat\sigma^2 = \frac{6\,M_p}{\xi_0}, \quad \hat\omega_\mu = \omega_\mu - \frac{1}{q}\partial_\mu \ln\sigma^2; \quad \text{use } \tilde R = R - 3q D_\mu\omega^\mu - \frac{3}{2}q^2\omega_\mu\omega^\mu$$

$$L_2 = \sqrt{\hat g}\,\Big\{-\frac{1}{2}\,M_p^2\,\hat R - \frac{3\,M_p^4}{2\,\xi_0} + \frac{3}{4}\,q^2\,M_p^2\,\hat\omega_\mu\,\hat\omega^\mu - \frac{1}{4}\,\hat F_{\mu\nu}^2\Big\}.$$

⇒ Einstein-Proca action for massive ω_μ which "absorbed" the dilaton $\ln\sigma$ field! mass of $\omega_\mu \propto q\,M_p$.

⇒ Gravitational Stueckelberg mechanism, massless $\sigma + \omega_\mu \to$massive ω_μ. no ghost! # dof=3 conserved!

⇒ no flat direction/dilaton left at low energies! Original Weyl gravity physically viable.

• The fate of the dilaton: Adding matter $\qquad\qquad$ ϕ- higgs-like field.

$$L_2 = \sqrt{g}\,\Big\{\frac{\xi_0}{4!}\,\tilde R^2 - \frac{1}{4}\,F_{\mu\nu}^2 - \frac{1}{12}\,\xi\,\phi^2\,\tilde R + \frac{1}{2}(\tilde D_\mu\phi)^2 - \frac{\lambda}{4!}\phi^4\Big\}, \qquad \text{linearise:} \quad \tilde R^2 \rightarrow -2\sigma^2\,\tilde R - \sigma^4$$

$$= \sqrt{g}\,\Big\{\frac{1}{4!}\,(\xi_0\,\sigma^2 + \xi\,\phi^2)\,\tilde R - \frac{1}{4}\,F_{\mu\nu}^2 + \frac{1}{2}(\tilde D_\mu\phi)^2 - \frac{1}{4!}\,(\lambda\,\phi^4 + \xi_0\,\sigma^4)\Big\}, \qquad \rho^2 = \frac{1}{6}\,(\xi_0\sigma^2 + \xi\,\phi^2)$$

• Weyl gauge transformation (b): "gauge fixing": $\qquad \Omega = \frac{\rho^2}{M_p^2}, \quad \hat\rho = M_p, \quad \hat\omega_\mu = \omega_\mu - \frac{1}{q}\ln\rho^2$

Einstein-Proca action for ω_μ :

$$L_2 = \sqrt{\hat g}\,\Big\{ - \frac{1}{2}\,M_p^2\,\hat R + \frac{3}{4}M_p^2\,q^2\,\hat\omega_\mu\hat\omega^\mu - \frac{1}{4}\hat F_{\mu\nu}^2 + \frac{1}{2}(\hat{\hat D}_\mu\hat\phi)^2 - V\Big]\Big\},$$

$$V = \frac{3M_p^4}{2\,\xi_0}\Big[1 - \frac{\xi\hat\phi^2}{6\,M_p^2}\Big]^2 + \frac{\lambda}{4!}\,\hat\phi^4.$$

$\Rightarrow$ No dilaton $\ln\sigma$ left (eaten by ω_μ). Gravitational higgs mechanism. Also $m_\phi^2 = (-\xi/\xi_0)\,M_p^2$.

$\Rightarrow$ Fixed point for ξ? The limit $\xi \rightarrow 0 \Rightarrow S_v \times S_h \Rightarrow \beta_\xi \propto \xi$? not $(\xi + 1/6)$! Quantum corrections?

- Conclusions

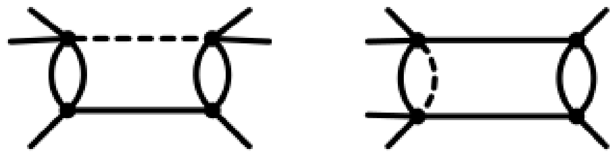
I. Flat space:

- Quantum SI: V in $\lambda\phi^4$ (3loop) & SM (1 loop) spontaneous breaking (SB)
- classical hierarchy of: higgs vev $\ll$ dilaton vev $\Rightarrow$ quantum stable
- SI effective operators, suppressed by dilaton
- all scales from fields' vevs, result not a DR artefact

II. Curved space:

- Generate $M_{\text{Planck}} \sim$ dilaton vev $\Rightarrow$ Weyl gravity/gauge symmetry, no ghost.
- Stueckelberg: dilaton "eaten" by Weyl "photon" ω_μ which becomes massive $m \sim q M_p$
- "Gauge fixing" Weyl gravity $=$ Einstein-Proca for ω_μ + cc > 0 after SB
- Adding matter: $\xi\phi^2\tilde{R}$: $\rightarrow$ EWSB; Fixed point for ξ? Next: m_h quantum stability?

$\epsilon \frac{\tilde{\sigma}}{\langle\sigma\rangle} \phi^4$



- Three-loop SI potential

Counterterm: $\delta L_3 = \frac{1}{2}\,\delta_\phi^{(3)}\,(\partial_\mu\phi)^2 - \mu^{2\epsilon}\Big(\frac{1}{4!}\,\delta_\lambda^{(3)}\,\lambda\,\phi^4 + \frac{1}{6}\,\delta_{\lambda_6}^{(3)}\,\lambda_6\,\frac{\phi^6}{\sigma^2} + \frac{1}{8}\,\delta_{\lambda_8}^{(3)}\,\lambda_8\,\frac{\phi^8}{\sigma^4}\Big)$ [Monin 2015]

$$\delta_\phi^{(3)} = -\frac{\lambda^3}{4\kappa^3}\Big(\frac{1}{6\epsilon^2} - \frac{1}{12\epsilon}\Big), \qquad \delta_{\lambda_6}^{(3)} = \frac{3}{2}\frac{\lambda^4}{\lambda_6\,\kappa^3\,\epsilon}, \qquad \delta_{\lambda_8}^{(3)} = \frac{275}{864}\frac{\lambda^4}{\lambda_8\,\kappa^3\,\epsilon.}$$

So $\gamma_\phi^{(3)} = \lambda^3/(16\kappa^3)$. With $Z_X = 1 + \delta_X$ and $\lambda_6^B = \mu^{2\epsilon}(\sigma)\lambda_6 Z_{\lambda_6} Z_\phi^{-3} Z_\sigma$ and $(d/d\ln z)\,\lambda_6^B = 0$,

$$\beta_{\lambda_6} \;=\; \frac{\lambda^2\,\lambda_6}{2\kappa^2} + \frac{\lambda^3}{\kappa^3}\Big(9\lambda - \frac{3}{8}\lambda_6\Big), \qquad \text{(similar } \beta_{\lambda_8}\text{).}$$

Callan-Symanzik: $V^{(3)}$ ["usual" with $\mu \to \sigma$]$+V^{(3,n)}$ [new]:

$$\frac{\partial V^{(3)}}{\partial \ln z} + \beta_\lambda^{(1)}\frac{\partial V^{(2)}}{\partial\lambda} + \beta_\lambda^{(2)}\frac{\partial V^{(1)}}{\partial\lambda} + \beta_\lambda^{(3)}\frac{\partial V}{\partial\lambda} + \gamma_\phi^{(2)}\,\frac{\partial V^{(1)}}{\partial \ln\phi} + \gamma_\phi^{(3)}\,\frac{\partial V}{\partial\ln\phi} = \mathcal{O}(\lambda_j^5).$$

$$\frac{\partial V^{(3,n)}}{\partial \ln z} + \beta_{\lambda_j}^{(1)}\frac{\partial V^{(2,n)}}{\partial\lambda_j} + \beta_{\lambda_j}^{(3,n)}\frac{\partial V}{\partial\lambda_j} = \mathcal{O}(\lambda_j^5), \qquad \lambda_j = \lambda, \lambda_6, \lambda_8.$$

• Three-loop SI potential: Integrate Callan-Symanzik:

$$U = V + \Delta V + V^{(3)} + V^{(3,n)}, \quad \Delta V = \lambda_6 \frac{\phi^6}{\sigma^2} + \lambda_8 \frac{\phi^8}{\sigma^4}$$

$$V^{(3)} = \frac{\lambda^4 \phi^4}{\kappa^3} \Big\{ \mathcal{Q} + \Big(\frac{97}{128} + \frac{9}{64} A_0 + \frac{\zeta[3]}{4}\Big) \overline{\ln} \frac{V_{\phi\phi}}{(z\sigma)^2} - \frac{31}{96} \overline{\ln}^2 \frac{V_{\phi\phi}}{(z\sigma)^2} + \frac{9}{64} \overline{\ln}^3 \frac{V_{\phi\phi}}{(z\sigma)^2} \Big\}.$$

$$V^{(3,n)} = \frac{\lambda^3}{2\,\kappa^3} \phi^4 \Big\{ \Big(27\lambda - \frac{\lambda_6}{2}\Big) \frac{\phi^2}{8\,\sigma^2} + \Big(\frac{401\,\lambda}{72} - \lambda_8\Big) \frac{\phi^4}{16\,\sigma^4} \Big\} \overline{\ln} \frac{V_{\phi\phi}}{(z\sigma)^2}. \qquad V_{\phi\phi} = \frac{\lambda}{2} \phi^2.$$

⇒ more non-polynomial operators at higher orders. Vanish for $\phi \ll \sigma$; $\ln \sigma$ left.

$\beta_\lambda^{(3)}$: unchanged (as if no dilaton; μ=const). Higher loops?

• some conclusions so far:

⇒ only terms suppressed by σ. For large σ enahnced shift symmetry $S_v \times S_h$ restored.

⇒ No terms such as: $\lambda\,\phi^2\sigma^2 = \lambda\,\langle\sigma\rangle^2\,\phi^2 + \cdots$. No tunning of λ for large σ. More scalars?

- Conformal group: Generators (15): P_μ, $M_{\mu\nu}$, D, K_μ. Conformal transformation leave angles invariant.

$$\text{infinitesimal transf:} \qquad \delta = -i\Big[b^\mu\, P_\mu - \frac{1}{2}\omega^{\mu\nu}\, M_{\mu\nu} - \rho\, D + c^\mu\, K_\mu\Big]$$

$$P_\mu = i\,\partial_\mu, \qquad M_{\mu\nu} = i\,(x_\mu\partial_\nu - x_\nu\,\partial_\mu) \qquad D = i\,x^\mu\partial_\mu, \qquad K_\mu = i\,(2\,x_\mu\,x^\nu - \delta^\nu_\mu\,x^2)\,\partial_\nu$$

for translation (P_μ), Lorentz ($M_{\mu\nu}$), dilatation (D), conformal (K_μ), giving: $x'^\mu = x^\mu + \epsilon\, f^\mu$

$$f_\mu \;=\; b_\mu, \quad f_\mu = \omega_{\mu\nu}\,x^\nu, \quad f_\mu = \rho\,x_\mu, \quad f_\mu = c_\nu\,(2\,x_\mu\,x^\nu - \delta^\nu_\mu\,x^2).$$

sol. to $\partial_\mu f_\nu + \partial_\nu f_\mu = (g_{\mu\nu}/2)\partial_\lambda f^\lambda$. $J^\mu = \theta^{\mu\nu}\, f_\nu(x)$ conserved. $\partial_\mu J^\mu = f_\nu\,\partial_\mu\theta^{\mu\nu} + (1/4)\,\theta^\mu_\mu\;\partial_\lambda f^\lambda = 0$.
Goldstone: $Q(t) = \int d\vec{x} J^0(\vec{x},t)$, and $0 = \int d\vec{x}[\partial_\mu J^\mu, \phi(0)] = \partial_0[Q(t),\phi(0)]$ or $\langle 0|[Q(t),\phi(0)]|0\rangle = \mathrm{c} \neq 0$.

$$\langle 0|[\theta^{\mu\nu}(x),\phi(y)]|0\rangle = \frac{-d}{3}\langle 0|\phi(0)|0\rangle\partial^\mu\partial^\nu D(x-y),$$

$$\langle 0|[D,\phi(y)]|0\rangle = d\,\langle 0|\phi(y)|0\rangle, \qquad \langle 0|[K^\mu,\phi(y)]|0\rangle = 2\,y^\mu\,d\,\langle 0|\phi(y)|0\rangle,$$

• The fate of the dilaton: Gravitational Stueckelberg mechanism:

A Weyl gauge invariant action:

$$L = \sqrt{g}\,\Big[-\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}\,(\tilde{D}_\mu\sigma)^2 + \cdots \Big], \qquad F_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu$$

$$\tilde{D}_\mu\sigma \;=\; (\partial_\mu - q/2\,\omega_\mu)\,\sigma = (-q/2)\,\sigma\,\big[\,\omega_\mu - (1/q)\,\partial_\mu \ln\sigma^2\,\big].$$

"gauge fixing" transformation: $\Omega^2 = \dfrac{\sigma^2}{M^2}, \quad \hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \quad \hat{\sigma} = \dfrac{1}{\Omega}\,\sigma \quad \hat{\omega}_\mu = \omega_\mu - \dfrac{1}{q}\,\partial_\mu \ln\Omega^2$

$\rightarrow$ Proca action

$$L = \sqrt{\hat{g}}\,\Big[-\frac{1}{4}\,\hat{F}_{\mu\nu}^2 + \frac{q^2}{8}\,M^2\hat{\omega}_\mu\hat{\omega}^\mu + \cdots \Big].$$

- spontaneous breaking; ω_μ has eaten the scalar $\ln\sigma$ (note $\ln\sigma \rightarrow \ln\sigma - \ln\Omega$)
- number of d.o.f. is conserved (=3).

• The fate of the dilaton: Weyl quadratic gravity $\to$ Einstein gravity + massive ω_μ [DG 1812.08613]

$$L_1 = \sqrt{g}\,\Big[\frac{\xi_0}{4!}\,\tilde R^2 - \frac{1}{4}\,F_{\mu\nu}^2\Big], \quad \xi_0>0, \qquad \text{original Weyl action (1918)}$$

$$= \sqrt{g}\,\Big[\frac{\xi_0}{4!}\big(-2\,\sigma^2\,\tilde R - \sigma^4\big) - \frac{1}{4}\,F_{\mu\nu}^2\Big] \qquad \text{eom: } \sigma^2 = -\tilde R; \quad \text{dilaton: } \ln\sigma \to \ln\sigma - \ln\Omega$$

Use $\tilde R = R - 3q\, D_\mu \omega^\mu - 3/2\, q^2\, \omega_\mu\omega^\mu$ then, in Riemann language:

$$L_1 = \sqrt{g}\,\Big\{ -\frac{\xi_0}{2}\,\Big[\frac{1}{6}\sigma^2\, R + (\partial_\mu\sigma)^2\Big] - \frac{\xi_0}{4!}\sigma^4 + \frac{q^2}{8}\,\xi_0\,\sigma^2\,\big(\omega_\mu - 1/q\,\partial_\mu \ln\sigma^2\big)^2 - \frac{1}{4}\,F_{\mu\nu}^2\Big\}.$$

• gauge transf (b) "gauge fixing!" $\quad \Omega = \dfrac{\xi_0\,\sigma^2}{6M_p^2} \Rightarrow \hat g_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \hat\sigma^2 = \dfrac{6\,M_p}{\xi_0}, \quad \hat\omega_\mu = \omega_\mu - \dfrac{1}{q}\partial_\mu \ln\sigma^2$

$$L_1 = \sqrt{\hat g}\,\Big\{ -\frac{1}{2}\,M_p^2\,\hat R - \frac{3\,M_p^4}{2\xi_0} + \frac{3}{4}\,q^2\,M^2 \hat\omega_\mu\,\hat\omega^\mu - \frac{1}{4}\,\hat F_{\mu\nu}^2\Big\}.$$

$\Rightarrow$ Einstein-Proca action for (massive) ω_μ which has "eaten" $\ln\sigma$ field! no ghost! # dof conserved!

cons. current $\quad \partial^\alpha(F_{\alpha\mu}\sqrt{g}) + \dfrac{1}{2}\sqrt{g}\,\xi_0\,\phi_0\,\Big[\partial_\mu - \dfrac{q}{2}\,\omega_\mu\Big]\,\phi_0 = 0$