ABSENCE OF CP VIOLATION IN YANG–MILLS THEORY WITH FERMIONS

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Outline

- Introduction
- Green's functions
- Correlation functions & effective operators
- Conclusions


INTRODUCTION

The theta-parameter in Yang–Mills theory

Consider SU(2) Yang–Mills with $N_f = 1$ massive fermions
(can be generalized to SU(N), $N_f > 1$):

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(\mathrm{i}\not{D} - m\mathrm{e}^{\mathrm{i}\alpha\gamma^5})\psi + \underbrace{\frac{1}{16\pi^2}\theta\,\mathrm{tr}F_{\mu\nu}\tilde{F}^{\mu\nu}}_{\text{toplogical term}}$$

$$D_\mu = \partial_\mu - \mathrm{i}A^a_\mu\frac{\tau^a}{2} = \partial_\mu + A_\mu \qquad [D_\mu, D_\nu] = -\mathrm{i}F_{\mu\nu}$$

$$F_{\mu\nu} = F^a_{\mu\nu}\frac{\tau^a}{2} \qquad \tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

Theta term (topological term) is CP-odd
$\bar{\psi}\mathrm{i}\gamma^5 m\sin\alpha\,\psi$ is CP-odd

Extended solutions in Euclidean space

Theta term is a total divergence

$$\frac{1}{4}\mathrm{tr}F_{\mu\nu}\tilde{F}_{\mu\nu} = \partial_\mu K_\mu \qquad\qquad K_\mu = \epsilon_{\mu\nu\alpha\beta}\mathrm{tr}\left[\frac{1}{2}A_\nu\partial_\alpha A_\beta + \frac{1}{3}A_\nu A_\alpha A_\beta\right]$$

$\rightarrow$ Equivalent to a surface term, i.e. the flux of the current through the boundary of the integration volume

So does it vanish?

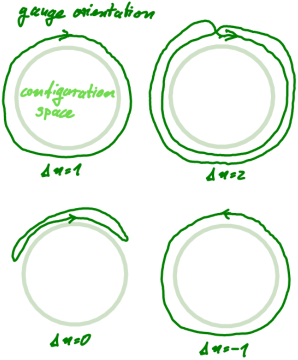
Cf. anti-instanton: $A_\mu{}^u{}_v = -\frac{\sigma_{\mu\nu}{}^u{}_v x_\nu}{x^2+\rho^2}$

Surface term decays as $1/|x|^3 \rightarrow$ surface integral does not need to vanish

For $x^2 \to \infty$, the field becomes a pure gauge:

$$A_\mu \to -\frac{\mathrm{i}}{g}(\partial_\mu \Omega)\Omega^{-1} \quad \text{where} \quad \Omega \in \mathrm{SU}(2)$$

$$K_\mu \to \frac{1}{6}\varepsilon_{\mu\nu\lambda\rho}\mathrm{tr}[(\Omega^{-1}\partial_\nu\Omega)(\Omega^{-1}\partial_\lambda\Omega)(\Omega^{-1}\partial_\rho\Omega)]$$



$$\Delta n = \frac{1}{16\pi^2}\int \mathrm{d}^4x F_{\mu\nu}\tilde{F}_{\mu\nu} = \frac{1}{4\pi^2}\oint_{S^3} \mathrm{d}^3\sigma K_\perp$$

Integrand is a Haar measure and maps $S^3 \to S^3$

(Anti-)instanton is a configuration with winding number $\Delta n = (-)1$

Theta term contributes to the action though being a total derivative

Theta vacuum

Now consider intitial and final states, taking $x_4 \to \pm\infty$
$\to$ Pure gauge configurations on these surfaces, with

$$\Delta n = \frac{1}{16\pi^2}\int \mathrm{d}^4 x F_{\mu\nu}\tilde{F}_{\mu\nu} = n_\infty - n_{-\infty} \quad \text{gauge invariant}$$

$$n_{\pm\infty} = \frac{1}{4\pi^2}\int\limits_{x^4=\pm\infty} \mathrm{d}^3\sigma K_\perp \qquad \textbf{not} \text{ gauge invariant}$$

Gauge transformations Ω change $n_{\pm\infty}$ by same number of integer units

Minkowskian boundary conditions fixed by prevacua: $\begin{array}{r} n_{-\infty} \to |n\rangle \\ n_\infty \to \langle n| \end{array}$

Gauge invariant (up to phase) state $|\mathrm{vac}\rangle = \sum_n \mathrm{e}^{in\theta}|n\rangle$
Alternatively, set $|\mathrm{vac}\rangle = \sum_n |n\rangle$ and absorb θ in topological term

Consequence: In the path integral, sum over all topological sectors Δn, weigh these by $\exp(\mathrm{i}\Delta n\theta)$

- Summation necessary because boundary conditions set at $t = \pm\infty$
- Not how we usually impose boundary conditions on the path integral
- Neither a natural requirement in Euclidean space
- Alternatively: Cluster-decompostion argument

Cluster-decompostion argument

Consider expectation value of an operator $\mathcal{O}$ in spacetime volume Ω

$$\langle\mathcal{O}\rangle_\Omega = \frac{\sum\limits_{\Delta n=-\infty}^{\infty} f(\Delta n) \int\limits_{\Delta n} \mathcal{D}\phi\, \mathcal{O}\, \mathrm{e}^{-S_\Omega[\phi]}}{\sum\limits_{\Delta n=-\infty}^{\infty} f(\Delta n) \int\limits_{\Delta n} \mathcal{D}\phi\, \mathrm{e}^{-S_\Omega[\phi]}}$$

Asssume $\Delta n(\Omega) = \Delta n_1(\Omega_1) + \Delta n_2(\Omega_2)$

Factorize path integral into volume contributions:

$$\langle\mathcal{O}_1\rangle_\Omega = \frac{\sum\limits_{\Delta n_1=-\infty}^{\infty} \sum\limits_{\Delta n_2=-\infty}^{\infty} f(\Delta n_1 + \Delta n_2) \int\limits_{\Delta n_1} \mathcal{D}\phi\, \mathcal{O}_1\, \mathrm{e}^{-S_{\Omega_1}[\phi]} \int\limits_{\Delta n_2} \mathcal{D}\phi\, \mathrm{e}^{-S_{\Omega_2}[\phi]}}{\sum\limits_{\Delta n_1=-\infty}^{\infty} \sum\limits_{\Delta n_2=-\infty}^{\infty} f(\Delta n_1 + \Delta n_2) \int\limits_{\Delta n_1} \mathcal{D}\phi\, \mathrm{e}^{-S_{\Omega_1}[\phi]} \int\limits_{\Delta n_2} \mathcal{D}\phi\, \mathrm{e}^{-S_{\Omega_2}[\phi]}}$$

Independence of $\langle\mathcal{O}_1\rangle_\Omega$ from the fluctuations in Ω_2 is achieved if the contributions from Ω_2 cancel:

$$f(\Delta n_1 + \Delta n_2) = f(\Delta n_1) f(\Delta n_2) \Rightarrow f(\Delta n) = \mathrm{e}^{\mathrm{i}\theta\Delta n}$$

Fermions & CP violation

Add fermion ψ in fundamental representation of SU(2)
Mass: $\bar{\psi} m \exp(\mathrm{i}\alpha\gamma^5)\psi$

Chiral transformation of fermion field $\rightarrow$ rephasing of θ, α: [Fujikawa (1979,80)]

$$\psi \rightarrow \mathrm{e}^{\mathrm{i}\beta\gamma_5}\psi\,, \quad \theta \rightarrow \theta + 2\beta\,, \quad \alpha \rightarrow \alpha - 2\beta$$

Massless fermion $\rightarrow$ no CP violation

Massive fermion $\rightarrow$ can shuffle phases forth and back between topological term and fermion mass but **cannot** remove CP-odd phase $\alpha + \theta$ in general

Standard picture: CP-violating effects mediated by instantons and can be described by the effective 't Hooft vertex: ['t Hooft (1976,86)]:

$$\mathcal{L} \rightarrow \mathcal{L} - \Gamma_{N_f} \mathrm{e}^{\mathrm{i}\theta} \prod_{j=1}^{N_f} (\bar{\psi}_j P_{\mathrm{L}} \psi_j) - \Gamma_{N_f} \mathrm{e}^{-\mathrm{i}\theta} \prod_{j=1}^{N_f} (\bar{\psi}_j P_{\mathrm{R}} \psi_j),$$

where Γ_{N_f} is a coefficient

Claim:
Deriving the fermion correlation functions with boundary conditions of vanishing physical fields from the path integral leads to:
[Ai, Cruz, BG, Tamarit (2020)]

$$\mathcal{L} \to \mathcal{L} - \Gamma_{N_f} \mathrm{e}^{-\mathrm{i}\bar{\alpha}} \prod_{j=1}^{N_f} (\bar{\psi}_j P_{\mathrm{L}} \psi_j) - \Gamma_{N_f} \mathrm{e}^{\mathrm{i}\bar{\alpha}} \prod_{j=1}^{N_f} (\bar{\psi}_j P_{\mathrm{R}} \psi_j) \qquad \bar{\alpha} = \sum_j^{N_f} \alpha_j$$

Effective operator inferred from
Green's functions $\to$ correlation functions


GREEN'S FUNCTIONS
HIMMELSLEITER

Euclidean Green's function

Euclidean Green's function $S^{\mathrm{E}}(x^{\mathrm{E}}, x^{\mathrm{E}\prime})$ satisfies

$$(\not{D}^{\mathrm{E}} + m_{\mathrm{R}} + \mathrm{i}\gamma^5 m_{\mathrm{I}}) S^{\mathrm{E}}(x^{\mathrm{E}}, x^{\mathrm{E}\prime}) = \delta^4(x^{\mathrm{E}} - x^{\mathrm{E}\prime})$$

Construct S^{E} from the spectral sum in the massless limit

Spectrum:

$$\not{D}^{\mathrm{E}} \hat{\psi}^{\mathrm{E}}_{\lambda} = \left(\not{\partial}^{\mathrm{E}} + \gamma^{\mathrm{E}}_m A^{\mathrm{E}}_m \right) \hat{\psi}^{\mathrm{E}}_{\lambda} = \lambda^{\mathrm{E}} \hat{\psi}^{\mathrm{E}}_{\lambda}$$

$\longrightarrow$

$$S^{\mathrm{E}}(x^{\mathrm{E}}, x^{\mathrm{E}\prime}) = \sum\!\!\!\!\!\!\!\int_{\lambda^{\mathrm{E}}} \frac{\hat{\psi}^{\mathrm{E}}_{\lambda}(x^{\mathrm{E}}) \hat{\psi}^{\mathrm{E}\dagger}_{\lambda}(x^{\mathrm{E}\prime})}{\lambda^{\mathrm{E}}}$$

Since the Euclidean Dirac operator $\not{D}^{\mathrm{E}}$ is anti-Hermitian, its eigenfunctions can readily be assumed to be orthonormal.

Deal with small masses once spectrum is analyzed

Fermion zero-modes

Spectral sum for $m = 0$ is ill-defined because of the fermionic zero mode $\lambda^{\mathrm{E}} = 0$ in the instanton background

Euclidean index theorem: Winding number equals difference between number of right-handed and left-handed zero modes
$\rightarrow$ One left (right)-handed zero-mode for $\Delta n = -1$ ($\Delta n = 1$)

Left-handed zero mode ['t Hooft (1976)]

$$\hat{\psi}^{\mathrm{E}}_{0\mathrm{L}}(x^{\mathrm{E}}) = \begin{pmatrix} \chi^{\mathrm{E}}_0(x^{\mathrm{E}}) \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix}, \quad \text{where} \quad \chi^{\mathrm{E}}_0(x^{\mathrm{E}}) = \frac{\varrho u}{\pi \left[\varrho^2 + (x^{\mathrm{E}})^2\right]^{\frac{3}{2}}}, \quad u^{\alpha b} = \varepsilon^{\alpha b}$$

Now include mass @ first order in perturbation theory ($\Delta n = -1$ background) [Shifman, Vainshtein, Zakharov (1979)]

$$S^{\mathrm{E}}(x^{\mathrm{E}}, x^{\mathrm{E}\prime}) = \frac{\hat{\psi}^{\mathrm{E}}_0(x^{\mathrm{E}})\hat{\psi}^{\mathrm{E}\dagger}_0(x^{\mathrm{E}\prime})}{m\mathrm{e}^{-\mathrm{i}\alpha}} + \sum_{\lambda^{\mathrm{E}} \neq 0} \frac{\hat{\psi}^{\mathrm{E}}_\lambda(x^{\mathrm{E}})\hat{\psi}^{\mathrm{E}\dagger}_\lambda(x^{\mathrm{E}\prime})}{\lambda^{\mathrm{E}}}$$

For $\alpha \neq 0$ and arbitrary m, can use linear combinations of $\hat{\psi}^{\mathrm{E}}_{\lambda}$ and $\gamma^5 \hat{\psi}^{\mathrm{E}}_{\lambda}$ as solutions to the eigenvalue problem in the case of general complex masses
Eigenvalues are then given by

$$\xi^{\mathrm{E}}_{\pm}(\lambda^{\mathrm{E}}) = m_{\mathrm{R}} \pm \sqrt{(\lambda^{\mathrm{E}})^2 - m_{\mathrm{I}}^2}$$

$\rightarrow$ No perturbative approximation needed if full massless spectrum is known

Continuation to Minkowski spacetime

Analytic continuation:

$$x_4 \to \mathrm{e}^{-\mathrm{i}(\vartheta - \frac{\pi}{2})} t$$

$t \in \mathbb{R}$ $\vartheta = \pi/2$: Euclidean metric $\vartheta = 0^+$: Minkowski metric

Continuation of Dirac operator:

$$\begin{aligned} \not{D}^{\mathrm{E}} =& (\not{\partial}^{\mathrm{E}}_m + \gamma^{\mathrm{E}}_m A^{\mathrm{E}}_m) \\ & \to \left(-\mathrm{i} \frac{\partial}{\partial x^0} \gamma^{\mathrm{E}}_4 + \vec{\gamma}^{\mathrm{E}} \cdot \nabla + \gamma^{\mathrm{E}}_4 A^{\mathrm{E}}_4(\vec{x}, x_4 = \mathrm{i}x^0) + \vec{\gamma}^{\mathrm{E}} \cdot \vec{A}^{\mathrm{E}}(\vec{x}, x_4 = \mathrm{i}x^0) \right) \\ & = -\mathrm{i} \left(\frac{\partial}{\partial x^0} \gamma^0 + \vec{\gamma} \cdot \nabla + \gamma^0 A_0(x^0, \vec{x}) + \vec{\gamma} \cdot \vec{A}(x^0, \vec{x}) \right) = -\mathrm{i} \not{D} \, , \end{aligned}$$

where $\vec{\gamma} \cdot \nabla \equiv \sum_i \gamma^i \partial_i$ and accordingly for $\vec{\gamma} \cdot \vec{A}$, $\gamma^0 = \gamma^{\mathrm{E}}_4$ and $\gamma^i = \mathrm{i} \gamma^{\mathrm{E}}_i$ for $i = 1, 2, 3$

Green's functions, as they are inverse Dirac operators, transform straightforwardly.

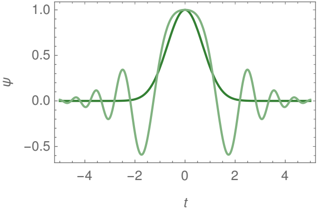
Continuation of the eigensystem

Issues of spectral representation in Minkowski spacetime:

- The operator $\mathrm{i}\not{D}\gamma^0$ in Minkowski spacetime is not of definite Hermiticity because of the complex gauge-field configuration in the analytically continued soliton background.
- The inner product $\int \mathrm{d}^4x\,\bar{\psi}_\xi(x)\psi_{\xi'}(x)$ is not positive definite.
- Zero-modes in anti-instanton (instanton) background are purely left (right)-handed. An operator breaking chiral symmetry—such as the effective instanton vertex—mixes left-and right chiral degrees of freedom. How does this play out in Minkowski spacetime?

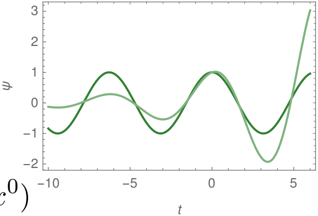
Determine continuation by behaviour in asymptotic, homogeneous spacetime region where the solutions go to either damped or oscillatory exponentials [Ai, BG, Tamarit (2019)]

Discrete modes—straightforward continuation as modes remain properly normalizable for $0 < \vartheta \leq \pi/2$:



$$\psi_n^\vartheta(x) = \psi_n^\vartheta(x^0, \vec{x}) = \sqrt{\mathrm{i}e^{-\mathrm{i}\vartheta}}\, \psi_n^{\mathrm{E}}(\vec{x}, x_4 = \mathrm{i}e^{-\mathrm{i}\vartheta} x^0)$$
$$\xi_n^\vartheta = -\, \xi_n^{\mathrm{E}}$$

Continuum modes—continue time **and** asymptotic k^0 (avoid blowup on one side, use asymptotic plane waves to label eigensystem)



$$\psi_{\{k\}}^\vartheta(x) = \psi_{\{k^0,\vec{k}\}}^\vartheta(x^0, \vec{x}) = \psi_{\{\vec{k}, -\mathrm{i}e^{\mathrm{i}\vartheta} k_0\}}^{\mathrm{E}}(\vec{x}, x_4 = \mathrm{i}e^{-\mathrm{i}\vartheta} x^0)$$
$$\xi_{\{k^0,\vec{k}\}}^\vartheta = -\, \xi_{\{\vec{k}, -\mathrm{i}e^{\mathrm{i}\vartheta} k_0\}}^{\mathrm{E}}$$

This eigensystem is orthonormal with respect to the following inner product:

$$(\psi_\xi^\vartheta, \psi_{\xi'}^\vartheta)_\vartheta = \int \mathrm{d}^4 x \, \tilde{\psi}_\xi^\vartheta(x) \, \psi_{\xi'}^\vartheta(x)$$

$$\tilde{\psi}_n^\vartheta(x^0, \vec{x}) = \sqrt{\mathrm{i}\mathrm{e}^{-\mathrm{i}\vartheta}} \, (\psi_n^{\mathrm{E}}(\vec{x}, x_4))^\dagger \Big|_{x_4 = \mathrm{i}\mathrm{e}^{-\mathrm{i}\vartheta} x^0} = \mathrm{i}\mathrm{e}^{-\mathrm{i}\vartheta} (\psi_n^\vartheta(x^0, \vec{x}))^\dagger \Big|_{x^0 \to -\mathrm{e}^{-2\mathrm{i}\vartheta} x^0}$$

$$\tilde{\psi}_{\{k^0, \vec{k}\}}^\vartheta(x^0, \vec{x}) = \left(\psi_{\{\vec{k}, k_4\}}^{\mathrm{E}}(\vec{x}, x_4)\right)^\dagger \Bigg|_{\substack{x_4 = \mathrm{i}\mathrm{e}^{-\mathrm{i}\vartheta} x^0 \\ k_4 = -\mathrm{i}\mathrm{e}^{\mathrm{i}\vartheta} k^0}} = \psi_{\{k^0, \vec{k}\}}^\vartheta(x^0, \vec{x})^\dagger \Bigg|_{\substack{x^0 \to -\mathrm{e}^{-2\mathrm{i}\vartheta} x^0 \\ k^0 \to -\mathrm{e}^{2\mathrm{i}\vartheta} k^0}}$$

Spectral representation in Minkowski spacetime

$$S^\vartheta(x, x') \equiv (\mathrm{i}\not{D}^\vartheta - m\mathrm{e}^{\mathrm{i}\alpha\gamma_5})^{-1}(x, x') = \sum\!\!\!\!\!\!\!\int_{\xi^\vartheta} \frac{1}{\xi^\vartheta} \, \psi_\xi^\vartheta(x) \tilde{\psi}_\xi^\vartheta(x')$$

$$= \sum_n \frac{1}{\xi_n^\vartheta} \, \psi_n^\vartheta(x) \tilde{\psi}_n^\vartheta(x') + \int \mathrm{d}^4 k \, \frac{1}{\xi_{\{k\}}^\vartheta} \psi_{\{k\}}^\vartheta(x) \tilde{\psi}_{\{k\}}^\vartheta(x')$$

Green's function in Minkowski spacetime

Application to zero mode in the $\eta = -1$ background gives

$$\psi_{0\mathrm{L}}(x^0, \vec{x}) \equiv \sqrt{\mathrm{i}}\, \varphi_{0\mathrm{L}}(x^0, \vec{x}) = \sqrt{\mathrm{i}}\, \psi^{\mathrm{E}}_{0\mathrm{L}}(\vec{x}, \mathrm{i}x^0)$$

where

$$\varphi_{0\mathrm{L}}(x) = \left(\begin{array}{c} \chi_0(x) \\ \left(\begin{array}{c} 0 \\ 0 \end{array} \right) \end{array} \right), \qquad \chi_0(x) = \frac{\varrho u}{\pi(\varrho^2 - x^2)^{\frac{3}{2}}}$$

Add contributions from far from the instanton [cf. Diakonov, Petrov (1986)]

$$\begin{aligned} \mathrm{i}S(x,x') &= \mathrm{i}S_{\mathrm{cont}}(x,x') + \frac{\varphi_{0\mathrm{L}}(x-x_0)\, \varphi^\dagger_{0\mathrm{L}}(x'-x_0)}{m\mathrm{e}^{-\mathrm{i}\alpha}} \\ &\approx \mathrm{i}S_{0\mathrm{inst}}(x,x') + \frac{\varphi_{0\mathrm{L}}(x-x_0)\, \varphi^\dagger_{0\mathrm{L}}(x'-x_0)}{m\mathrm{e}^{-\mathrm{i}\alpha}} \end{aligned}$$

$$\mathrm{i}S_{0\mathrm{inst}}(x,x') = (-\gamma^\mu\partial_\mu + \mathrm{i}m\mathrm{e}^{-\mathrm{i}\alpha\gamma^5}) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}p(x-x')} \frac{1}{p^2 - m^2 + \mathrm{i}\epsilon}$$

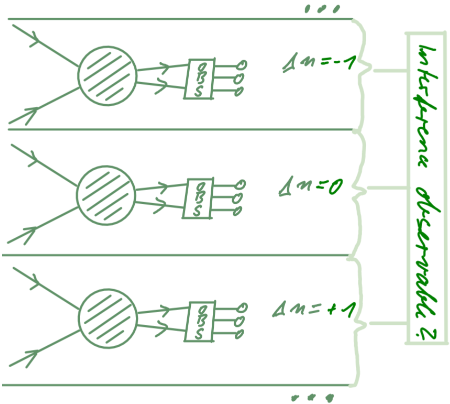
Green's function in n-instanton, $\bar{n}$-anti-instanton background

$$
\begin{aligned}
\mathrm{i}S_{n,\bar{n}}(x,x') \approx \mathrm{i}S_{0\mathrm{inst}}(x,x') &+ \sum_{\bar{\nu}=1}^{\bar{n}} \frac{\varphi_{0\mathrm{L}}(x-x_{0,\bar{\nu}})\varphi^{\dagger}_{0\mathrm{L}}(x'-x_{0,\bar{\nu}})}{m\mathrm{e}^{-\mathrm{i}\alpha}} \\
&+ \sum_{\nu=1}^{n} \frac{\varphi_{0\mathrm{R}}(x-x_{0,\nu})\varphi^{\dagger}_{0\mathrm{R}}(x'-x_{0,\nu})}{m\mathrm{e}^{\mathrm{i}\alpha}}
\end{aligned}
$$

Comments:

- For small masses, zero-modes dominate close to the core of the instantons, far away from the instanton the solution goes to the zero-instanton configuration
- ϑ-inner product explains how zero-mode contribution in Minkowski space can break χral symmetry
- **Alignment** of phase α between Lagrangian mass and instanton-induced χSB $\longrightarrow$ No indication of CP-violation here
- Perhaps expected—θ-vacuum has not entered calculation thus far
- Yet check out interference between different topological sectors Δn

Can interference between topological sectors be observed?



- Observer correponds to one or more legs of a correlation function (Feynman diagram)
- Reconstruct the state of the observer from amplitudes
- Works for each topological sector Δn separately (observer/system state evolves separately for each sector Δn)
- Possible to observe the interference between the topological sectors of different Δn? **Superobserver?**

Topological phases $\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$ appear globally for each topological sector. It is not clear how an observer made up of local quantum fields can access separate sectors neither should Δn be observable for $VT \to \infty$ to avoid collapse of the θ-vacuum.

Resolution: Turns out interferences are immaterial in the limit $VT \to \infty$


CORRELATION FUNCTIONS &
EFFECTIVE OPERATORS

Integrating out the fluctuations

Choose θ-vacuum in Minkowski spacetime as $|\mathrm{vac}\rangle = \sum_{n_{\mathrm{CS}}} |n_{\mathrm{CS}}\rangle$

Absorb CP-odd phase in topological term/fermion mass

Evaluate correlation and partition function first for **fixed** Δn

$$
\begin{aligned}
&\langle \psi(x)\bar\psi(x')\rangle_{\Delta n} \\
=&\sum_m {}_{\mathrm{out}}\langle m+\Delta n|\psi(x)\bar\psi(x')|m\rangle_{\mathrm{in}} = \sum_{\substack{\bar n, n\geq 0\\ n-\bar n=\Delta n}} \int \mathcal{D}A_{\bar n,n}\mathcal{D}\bar\psi\mathcal{D}\psi\,\psi(x)\bar\psi(x')\mathrm{e}^{\mathrm{i}S_{\bar n,n}} \\
=&\sum_{\substack{\bar n, n\geq 0\\ n-\bar n=\Delta n}} \frac{1}{\bar n! n!}\left(\prod_{\bar\nu=1}^{\bar n}\int_{VT}\mathrm{d}^4x_{0,\bar\nu}\mathrm{d}\Omega_{\bar\nu}J_{\bar\nu}\right)\left(\prod_{\nu=1}^{n}\int_{VT}\mathrm{d}^4x_{0,\nu}\mathrm{d}\Omega_{\nu}J_{\nu}\right)\mathrm{i}S(x,x') \\
&\times|\det(\mathrm{i}\partial\!\!\!/ - m\mathrm{e}^{\mathrm{i}\alpha\gamma^5})|\,(\det{}_{A=0})^{-1/2}\,\mathrm{e}^{-S_{\mathrm{E}}(\bar n+n)}\mathrm{e}^{-\mathrm{i}(\bar n-n)(\alpha+\theta)}(-\Theta\varpi)^{(\bar n+n)}
\end{aligned}
$$

$\mathrm{d}\Omega_\nu J_\nu$:	Zero modes & pertaining Jacobians
Θ,ϖ :	Reduced fermion & gauge/ghost determinants in instanton background
Note:	The explicit determinants in above formula are vacuum determinants.

Likewise, partition function:

$$Z_{\Delta n} = \sum_m {}_{\rm out}\langle m+\Delta n|m\rangle_{\rm in} = \sum_{\substack{\bar n, n\geq 0 \\ n-\bar n=\Delta n}} \int \mathcal{D}A_{\bar n,n}\mathcal{D}\bar\psi\mathcal{D}\psi\, \mathrm{e}^{\mathrm{i}S_{\bar n,n}}$$

$$= \sum_{\substack{\bar n, n\geq 0 \\ n-\bar n=\Delta n}} \frac{1}{\bar n! n!}\left(-\!\int\!\mathrm{d}\Omega\, J\, VT\, \Theta\, \varpi\, \mathrm{e}^{-S_{\rm E}}\right)^{(\bar n+n)} \times |\det(\mathrm{i}\not{\partial} - m\mathrm{e}^{\mathrm{i}\alpha\gamma^5})|\, (\det{}_{A=0})^{-1/2}\mathrm{e}^{-\mathrm{i}(\bar n-n)(\alpha+\theta)}$$

Integrate out locations of the instanton

$$\begin{aligned}
&\int_{VT} \mathrm{d}^4 x_{0,\bar\nu}\ \mathrm{i}S(x,x') \\
&\approx \int_{VT} \mathrm{d}^4 x_{0,\bar\nu} \left[\mathrm{i}S_{0\mathrm{inst}}(x,x') + \frac{\varphi_{0\mathrm{L}}(x-x_{0,\bar\nu})\varphi^\dagger_{0\mathrm{L}}(x'-x_{0,\bar\nu})}{m\mathrm{e}^{-\mathrm{i}\alpha}} + \cdots\right] \\
&= VT\left(\mathrm{i}S_{0\mathrm{inst}}(x,x') + \cdots\right) + m^{-1}\mathrm{e}^{\mathrm{i}\alpha} h(x,x') P_\mathrm{L}
\end{aligned}$$

Dots represent contributions from the zero modes of the (anti)-instantons whose centres were not integrated over

$h(x,x')$ is defined as a block-diagonal matrix (with two identical blocks):

$$h(x,x')P_\mathrm{L} = \int_{VT} \mathrm{d}^4 x_{0,\bar\nu}\, \varphi_{0\mathrm{L}}(x-x_{0,\bar\nu})\varphi^\dagger_{0\mathrm{L}}(x'-x_{0,\bar\nu})$$

$$h(x,x')P_\mathrm{R} = \int_{VT} \mathrm{d}^4 x_{0,\bar\nu}\, \varphi_{0\mathrm{R}}(x-x_{0,\bar\nu})\varphi^\dagger_{0\mathrm{R}}(x'-x_{0,\bar\nu})$$

$$\bar h(x,x') \equiv \frac{\int d\Omega\, h(x,x')}{\int d\Omega}$$

Integrating over all locations $\longrightarrow$ Correlation function for fixed Δn:

$$
\begin{aligned}
&\langle\psi(x)\bar\psi(x')\rangle_{\Delta n}\\
&=\sum_{\substack{\bar n,n\geq 0\\ n-\bar n=\Delta n}}\frac{1}{\bar n! n!}\left[\bar h(x,x')\left(\frac{\bar n}{m\mathrm{e}^{-\mathrm{i}\alpha}}P_{\mathrm{L}}+\frac{n}{m\mathrm{e}^{\mathrm{i}\alpha}}P_{\mathrm{R}}\right)(VT)^{\bar n+n-1}+\mathrm{i}S_{0\mathrm{inst}}(x,x')\,(VT)^{\bar n+n}\right]\\
&\qquad\qquad\times(\mathrm{i}\kappa)^{\bar n+n}(-1)^{n+\bar n}\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}\\
&=\left[\left(\mathrm{e}^{\mathrm{i}\alpha}I_{\Delta n+1}(2\mathrm{i}\kappa VT)P_{\mathrm{L}}+\mathrm{e}^{-\mathrm{i}\alpha}I_{\Delta n-1}(2\mathrm{i}\kappa VT)P_{\mathrm{R}}\right)\frac{\mathrm{i}\kappa}{m}\,\bar h(x,x')+I_{\Delta n}(2\mathrm{i}\kappa VT)\mathrm{i}S_{0\mathrm{inst}}(x,x')\right]\\
&\qquad\qquad\times(-1)^{\Delta n}\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}
\end{aligned}
$$

where $\mathrm{i}\kappa=\int\mathrm{d}\Omega\,J\,\Theta\,\varpi\,\mathrm{e}^{-S_{\mathrm{E}}}$ and $I_\alpha(x)$ is the modified Bessel function

Sum is dominated by particular value of $n\approx\bar n$:

$$
\langle n\rangle=\frac{\sum_{n=0}^{\infty}n\frac{(\alpha VT)^n}{n!}}{\sum_{n=0}^{\infty}\frac{(\alpha VT)^n}{n!}}=\alpha VT\,,\qquad\frac{\langle\Delta n\rangle}{\langle n\rangle}=\frac{1}{\sqrt{\alpha VT}}
$$

Cf. $\lim_{x\to\infty}I_{\Delta n}(\mathrm{i}x\,\mathrm{e}^{-\mathrm{i}0^+})/I_{\Delta n'}(\mathrm{i}x\,\mathrm{e}^{-\mathrm{i}0^+})=1$

$\longrightarrow$ No relative CP phase between mass and instanton induced breaking of χral symmetry—**alignment**

Correspondingly, partition function for fixed Δn:

$$Z_{\Delta n} = I_{\Delta n}(2\mathrm{i}\kappa VT)\,(-1)^{\Delta n}\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$$

Note: The topological phase $\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$ multiplies $\langle\psi(x)\bar{\psi}(x')\rangle_{\Delta n}$ and $Z_{\Delta n}$ entirely—not just the contributions induced by instantons.

Now, interfere over all Δn (i.e. sum over topological sectors) and see whether this makes a difference

Ordering of the limits

Δn only well-defined for $VT \to \infty$ (unless periodic boundary conditions)
$\longrightarrow$ Implies use of $\lim_{x\to\infty} I_{\Delta n}(\mathrm{i}x\,\mathrm{e}^{-\mathrm{i}0^+})/I_{\Delta n'}(\mathrm{i}x\,\mathrm{e}^{-\mathrm{i}0^+}) = 1$

Sum over topological sectors: interference

Partition function in θ-vacuum (recall phase resides in topological term):

$$Z = {}_{\rm out}\langle \mathrm{vac}|\mathrm{vac}\rangle_{\rm in} = \sum_{m,n} {}_{\rm out}\langle m|n\rangle_{\rm in} = \sum_{\Delta n=-\infty}^{\infty} \sum_m {}_{\rm out}\langle m+\Delta n|m\rangle_{\rm in} = \sum_{\Delta n=-\infty}^{\infty} Z_{\Delta n}$$

Fermion correlator

$$\langle \psi(x)\bar\psi(x')\rangle \equiv \frac{1}{Z}\, {}_{\rm out}\langle \mathrm{vac}|\psi(x)\bar\psi(x')|\mathrm{vac}\rangle_{\rm in}$$

$$= \frac{\sum\limits_{\Delta n=-\infty}^{\infty} \sum\limits_n {}_{\rm out}\langle n+\Delta n|\psi(x)\bar\psi(x')|n\rangle_{\rm in}}{\sum_{\Delta n=-\infty}^{\infty} Z_{\Delta n}} = \lim_{\substack{N\to\infty\\ N\in\mathbb{N}}} \lim_{VT\to\infty} \frac{\sum\limits_{\Delta n=-N}^{N} \langle \psi(x)\bar\psi(x')\rangle_{\Delta n}}{\sum_{\Delta n=-N}^{N} Z_{\Delta n}}$$

$$= \mathrm{i}S_{0\mathrm{inst}}(x,x') + \mathrm{i}\kappa \bar h(x,x') m^{-1} \mathrm{e}^{-\mathrm{i}\alpha\gamma^5} \quad \text{(same as for fixed } \Delta n)$$

Recall: $\mathrm{i}S_{0\mathrm{inst}}(x,x') = (-\gamma^\mu \partial_\mu + \mathrm{i}m\mathrm{e}^{-\mathrm{i}\alpha\gamma^5}) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}p(x-x')} \frac{1}{p^2-m^2+\mathrm{i}\epsilon}$

$\longrightarrow$ No relative CP-phase between mass and instanton term

Limits ordered the other way around

First sum over all Δn as well:

$$\sum_{\bar{n},n\geq 0} \frac{1}{\bar{n}!n!}\Big[\,\bar{h}(x,x')(\bar{n}\,m^{-1}\mathrm{e}^{\mathrm{i}\alpha}P_{\mathrm{L}} + n\,m^{-1}\mathrm{e}^{-\mathrm{i}\alpha}P_{\mathrm{R}})\,(VT)^{\bar{n}+n-1} + \mathrm{i}S_{0\mathrm{inst}}(x,x')\,(VT)^{\bar{n}+n}\Big] \times (-mi\kappa)^{\bar{n}+n}\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$$

$$= \left[-\left(\mathrm{e}^{-\mathrm{i}\theta}P_{\mathrm{L}} + \mathrm{e}^{\mathrm{i}\theta}P_{\mathrm{R}}\right)\frac{\mathrm{i}\kappa}{m}\bar{h}(x,x') + \mathrm{i}S_{0\mathrm{inst}}(x,x')\right]\mathrm{e}^{-2\mathrm{i}\kappa VT\cos(\alpha+\theta)}$$

$$Z \to \sum_{n,\bar{n}} \frac{1}{n!\bar{n}!}(-\mathrm{i}\kappa VT)^{\bar{n}+n}\mathrm{e}^{-\mathrm{i}(\bar{n}-n)(\alpha+\theta)} = \mathrm{e}^{-2\mathrm{i}\kappa VT\cos(\alpha+\theta)}$$

Then, $VT\to\infty$ trivial as VT-dependence cancels
$\longrightarrow$ Relative CP phase leading to CP-violating observables

However: The order of the limits is not a choice but dictated by the fact that boundary conditions for the theta-vacuum are imposed at $t=\pm\infty$.

Effective operators

Effective interactions in the theory with fermions (present analysis)
$\longrightarrow$ Effective operators in χral perturbation theory
$\longrightarrow$ Observables such as neutron EDM, $\eta' \to \pi\pi$

$VT \to \infty$ **before** $\sum_{\Delta n}$	$\sum_{\Delta n}$ **before** $VT \to \infty$
$\mathcal{L} \to \mathcal{L} - \bar{\psi}(x)\Gamma \mathrm{e}^{\mathrm{i}\alpha\gamma^5}\psi(x)$	$\mathcal{L} \to \mathcal{L} + \bar{\psi}(x)\Gamma \mathrm{e}^{-\mathrm{i}\theta\gamma^5}\psi(x)$
Alignment with $\bar{\psi} m \exp(\mathrm{i}\alpha\gamma^5)\psi$	Misaligned with $\bar{\psi} m \exp(\mathrm{i}\alpha\gamma^5)\psi$
No CP-violating observables	CP-violating observables

Note: both operators transform in compliance with χral anomaly:
$\psi \to \mathrm{e}^{\mathrm{i}\beta\gamma_5}\psi, \qquad \bar{\psi} \to \bar{\psi}\,\mathrm{e}^{\mathrm{i}\beta\gamma_5}, \qquad \alpha \to \alpha - 2\beta, \qquad \theta \to \theta + 2\beta$

N_f flavours: $\mathcal{L} \to \mathcal{L} - \Gamma_{N_f}\mathrm{e}^{-\mathrm{i}\bar{\alpha}}\prod_{j=1}^{N_f}(\bar{\psi}_j P_{\mathrm{L}}\psi_j) - \Gamma_{N_f}\mathrm{e}^{\mathrm{i}\bar{\alpha}}\prod_{j=1}^{N_f}(\bar{\psi}_j P_{\mathrm{R}}\psi_j)$

Can also compute more general correlation functions, where again factors $\sum_{\Delta n}(-1)^{\Delta n}\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$ cancel when taking $VT \to \infty$ first


CONCLUSIONS

Conclusions

- Have derived Green's functions for massive fermion in Yang–Mills theory (instantons) with general chiral phase, Euclidean & Minkowskian versions
  $\longrightarrow$ Alignment of χral phases from mass and instanton terms
- From these, have obtained correlation functions in theta vacuum
- In the infinite spacetime volume $VT \to \infty$, also the correlators show the alignment of χral phases.
- Thus, the absence CP-violating observables in QCD is explained without setting $\alpha + \theta = 0$ or requiring an extension of the Standard Model.