1. Homogeneous and isotropic universe

1.1 Friedmann-Lemaître-Robertson-Walker universe

Modern cosmology is based on the hypothesis that our universe is to a good approximation homogeneous and isotropic on sufficiently large length scales (Cosmological principle).

- Homogeneous → same everywhere
- Isotropic → same in all directions
- sufficiently large scales → $> \mathcal{O}(100\,\mathrm{Mpc})$ *

* $1\,\mathrm{pc} = 1$ parsec $= 3.0856 \times 10^{18}\,\mathrm{cm}$

e.g., - Distance from Sun to Galactic centre $\sim 10\,\mathrm{kpc}$
- Distance to Virgo cluster $\sim 20\,\mathrm{Mpc}$
- Visible universe $\sim \mathcal{O}(10\,\mathrm{Gpc})$

Evidence for large-scale homogeneity and isotropy:

① Cosmic microwave background, temperature $\sim 2.73\,\mathrm{K}$ in all directions, fluctuations $\delta T/T \sim 10^{-5}$.

② Galaxy distribution.

Homogeneity and isotropy imply maximally symmetric 3-spaces (3 translational and 3 rotational symmetries). This implies a general spacetime metric of the form:

$$\boxed{ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu = -dt^2 + a^2(t)\,\gamma_{ij}\,dx^i dx^j} \qquad \begin{matrix}\mu,\nu = 0,1,2,3\\ i,j = 1,2,3\end{matrix} \qquad (1.1.1)$$

FLRW metric

where $a(t)$ = scale factor (so that $a(t_2)/a(t_1)$ denotes how much space has expanded between t_1 and t_2), and

$$\boxed{\gamma_{ij}\,dx^i dx^j = \frac{dr^2}{1-Kr^2} + r^2 \underbrace{(d\theta^2 + \sin^2\theta\, d\phi^2)}_{d\Omega^2}} \qquad (1.1.2)$$

with $K=0$ flat, Euclidean

$K=+1$ spherical geometry of constant positive curvature

$K=-1$ hyperbolic geometry of constant negative curvature

Another <u>common</u> notation. Define

$$d\chi^2 \equiv \frac{dr^2}{1-Kr^2}$$

$$\Rightarrow \chi = \begin{cases} \text{arcsinh}\, r & , \quad K=-1 \\ r & \quad\;\; K=0 \end{cases} \Big\} \; 0 \le \chi < \infty \quad \text{open infinite}$$

$$\qquad\;\; \arcsin r \qquad K=+1 \quad 0 \le \chi < \pi \quad \text{closed finite}$$

$$(1.1.3)$$

Thus, (1.1.2) becomes

$$\gamma_{ij}\,dx^i dx^j = d\chi^2 + \begin{pmatrix} \sinh^2\chi \\ \chi^2 \\ \sin^2\chi \end{pmatrix} d\Omega^2 \quad \begin{matrix} K=-1 \\ K=0 \\ K=+1 \end{matrix} \qquad (1.1.4)$$

Observational evidence : flat spatial geometry ($K=0$) is preferred. (Boomerang 2000).

1.2 Matter/energy content

Matter/energy content is encoded in the stress-energy tensor $T^{\alpha\beta}$. Homogenity and isotropy require that

$$T^{0i} = 0 \quad , \quad T^{ij} = 0 \ (i \neq j) \qquad (1.2.1)$$

Thus, the only viable form is:

$$\boxed{T_{\mu\nu} = \begin{pmatrix} -\rho g_{00} & 0 \\ 0 & P g_{ij} \end{pmatrix}} \qquad (1.2.2)$$

where ρ = energy density and P = pressure in the rest frame of the coordinates (1.1.1), or the comoving frame.

ρ and P must be independent of x^i because of the homogenity and isotropy requirement, but can depend on t.

What matter/energy?

- Photons (mainly the cosmic microwave background).
- Atoms
- Dark matter (does not emit light but feels gravity)
- Gravitational wave background ??
- Background neutrinos (analogous to the CMB).
- Vacuum energy / dark energy.

In general:

$$T_{\mu\nu} \longrightarrow \sum_{\substack{i = \text{all} \\ \text{matter/energy} \\ \text{components}}} T_{\mu\nu}^{(i)} \qquad (1.2.3)$$

For each component i, local conservation of energy-momentum implies:

$$\nabla_\alpha T^{(i)\alpha}{}_{\beta} = 0$$

Covariant derivative

$$= \frac{\partial T^{(i)\alpha}{}_{\beta}}{\partial x^\alpha} + \Gamma^\alpha_{\gamma\alpha} T^{(i)\gamma}{}_{\beta} - \Gamma^\gamma_{\alpha\beta} T^{(i)\alpha}{}_{\gamma} \quad (1.2.4)$$

Christoffel symbols

Where:
$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}\left(\frac{\partial g_{\delta\beta}}{\partial x^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta}\right) \quad (1.2.5)$$

Combining FLRW metric (1.1.1) + stress-energy tensor (1.2.2) + (1.2.4):

$$\Rightarrow \boxed{\frac{d\rho_i}{dt} + 3\frac{\dot{a}}{a}(\rho_i + P_i) = 0} \quad (1.2.6)$$

Continuity equation

$$\dot{a} \equiv \frac{da}{dt}$$

It remains to specify an equation of state relating P_i and ρ_i for the fluid i. Define:

$$\boxed{w_i \equiv \frac{P_i}{\rho_i}} \quad (1.2.7)$$

equation of state parameter

Assuming a constant w_i:

$$(1.2.6) \Rightarrow \frac{d\rho_i}{dt} + 3(1+w_i)\frac{\dot{a}}{a}\rho_i = 0$$

$$\Rightarrow \boxed{\rho_i(t) \propto a^{-3(1+w_i)}} \quad (1.2.8)$$

① Nonrelativistic matter, e.g., dark matter, atoms: $w=0$

$$\Rightarrow \rho_m \propto a^{-3} \tag{1.2.9}$$

② Relativistic fluids (radiation), e.g., photons: $w=\frac{1}{3}$

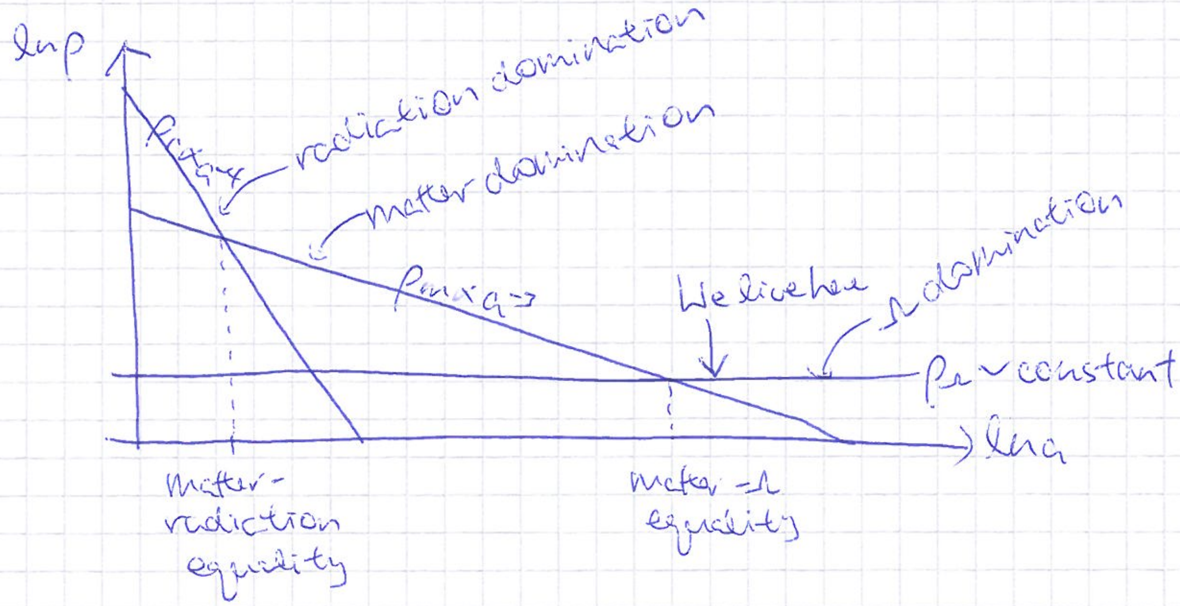
$$\Rightarrow \rho_r \propto a^{-4} \tag{1.2.10}$$

③ Vacuum energy: $w=-1$

$$\Rightarrow \rho_\Lambda \propto \text{constant} \tag{1.2.11}$$

As $a \to 0$, $\rho_r \gg \rho_m, \rho_\Lambda$

$a \to \infty$, $\rho_\Lambda \gg \rho_m, \rho_r$



1.3 Friedmann equation

An equation of motion for $a(t)$ derived from the Einstein equation:

Newton's constant

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \equiv G_{\mu\nu} = 8\pi G\, T_{\mu\nu}} \qquad (1.3.1)$$

Einstein tensor

① $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ Ricci tensor

② $R = R^{\mu}{}_{\mu}$ Ricci scalar (1.3.2)

③ $R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}_{\nu\beta,\mu} - \Gamma^{\alpha}_{\mu\beta,\nu} + \Gamma^{\delta}_{\beta\nu}\Gamma^{\alpha}_{\mu\delta} - \Gamma^{\delta}_{\beta\mu}\Gamma^{\alpha}_{\nu\delta}$ Riemann tensor

Note: (1.3.1) can be obtained by minimising the action:

$$S = S_{EH} + S_M$$

where:

$$S_{EH} = -\frac{1}{16\pi G}\int d^4x \sqrt{-g}\, R \qquad \text{Einstein-Hilbert action} \qquad (1.3.3)$$

$$S_M = \int d^4x \sqrt{g}\, \mathcal{L}_M \leftarrow \text{Lagrangian density for matter fields}$$

$$g \equiv \det(g_{\mu\nu})$$

Varying S with respect to $g^{\mu\nu}$, we find:

$$\delta S_{EH} = \frac{1}{16\pi G}\int d^4x \sqrt{g}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)\delta g^{\mu\nu} \qquad (1.3.4)$$

$$\delta S_M = -\frac{1}{2}\int d^4x \sqrt{g}\, \underbrace{\left[\frac{-2}{\sqrt{g}}\frac{\partial(\sqrt{g}\,\mathcal{L}_M)}{\partial g^{\mu\nu}}\right]}_{\equiv T_{\mu\nu} \text{ by definition}} \delta g^{\mu\nu}$$

By demanding $\delta S / \delta g^{\mu\nu} = 0$, we recover (1.3.1).

Using the FLRW metric (1.1.1) and the stress-energy tensor (1.2.2), we find:

① $R_{00} - \frac{1}{2} g_{00} R = 8\pi G T_{00}$

$$\Rightarrow \boxed{\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3} \sum_i \rho_i} \qquad (1.3.5)$$

Friedmann eqn

② $R_{ij} - \frac{1}{2} g_{ij} R = 8\pi G T_{ij}$

$$\Rightarrow 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = -8\pi G \sum_i P_i \qquad (1.3.6)$$

Note: we usually use (1.3.5) and the continuity eqn (1.2.6). (1.3.6) is redundant.

Another useful form:

$$(1.3.6) - (1.3.5) \Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \sum_i (\rho_i + 3P_i) \qquad (1.3.7)$$

acceleration.

Define the Hubble parameter:

$$\boxed{H(t) \equiv \frac{\dot{a}}{a}} \qquad (1.3.8)$$

Then, we can also write the Friedmann eqn (1.3.5) as:

$$\frac{K}{a^2 H^2} = \frac{\sum_i \rho_i(t)}{\rho_{crit}(t)} - 1 \qquad (1.3.9)$$

where: $\rho_{crit}(t) \equiv \frac{3H^2(t)}{8\pi G}$ critical density

i.e., if $\sum_i \rho_i(t) = \rho_{crit}(t)$, then $K=0 \Rightarrow$ flat spatial geometry.

Define the energy density parameters:

$$\boxed{\Omega_i \equiv \frac{\rho_i(t_0)}{\rho_{crit}(t_0)}} \qquad t_0 = \text{today} \qquad (1.3.10)$$

Then the Friedmann equation becomes: $a_0 \equiv a(t_0) = 1$ ← convention

$$H^2(t) = \underbrace{\frac{8\pi G}{3}\rho_{crit}(t_0)}_{H^2(t_0)}\left[\Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_r\left(\frac{a_0}{a}\right)^4 + \Omega_\Lambda\right] - \frac{K}{a^2}$$

$$= H^2(t_0)\left[\Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_r\left(\frac{a_0}{a}\right)^4 + \Omega_\Lambda + \Omega_K\left(\frac{a_0}{a}\right)^2\right] \qquad (1.3.11)$$

Where: $$\boxed{\Omega_K \equiv -\frac{K}{a_0^2 H^2(t_0)}} \qquad (1.3.12)$$

Current observations: $\Omega_m \sim 0.3$, $\Omega_\Lambda \sim 0.7$, $\Omega_r \sim 10^{-5}$

Komatsu et al. (WMAP7) $\quad |\Omega_K| \lesssim 0.01$

arxiv:1001.4538 $\quad H_0 \sim 70\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$

Solutions to (1.3.11)

① If radiation dominates the matter/energy content:

$$\left(\frac{\dot a}{a}\right)^2 \propto a^{-4} \Rightarrow \boxed{a \propto t^{1/2}} \qquad (1.3.13)$$

② If matter dominates:

$$\left(\frac{\dot a}{a}\right)^2 \propto a^{-3} \Rightarrow \boxed{a \propto t^{2/3}} \qquad (1.3.14)$$

③ If Λ dominates:

$$\frac{\dot a}{a} = \underbrace{H_0\sqrt{\Omega_\Lambda}}_{\text{constant}} \quad \Big\| \quad H_0 \equiv H(t_0) \Rightarrow \boxed{a \propto \exp(H_0\sqrt{\Omega_\Lambda}\, t)} \qquad (1.3.15)$$

1.4 Cosmological redshift

Using the invariant:

$$\underset{\text{4-momentum}}{\tilde{P}^\alpha} \tilde{P}_\alpha = g_{\alpha\beta} \tilde{P}^\alpha \tilde{P}^\beta = -m^2 \qquad (1.4.1)$$

and the geodesic equation:

$$\frac{d\tilde{P}^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma} \tilde{P}^\beta \tilde{P}^\gamma = 0 \qquad (1.4.2)$$

we can show that the proper / physical momentum of a particle measured by a comoving observer (comoving = at rest with the coordinates (1.1.1)) evolves as:

$$\boxed{|\vec{p}| \propto a^{-1}} \qquad (1.4.3)$$

For photons emitted at $t = t_e$ and observed at $t = t_0$, this means:

$$\boxed{\frac{|\vec{p}(t_e)|}{|\vec{p}(t_0)|} = \frac{E(t_e)}{E(t_0)} = \underbrace{\frac{\lambda(t_0)}{\lambda(t_e)}}_{\text{wavelength}} = \frac{a(t_0)}{a(t_e)} = (1+\underbrace{z}_{\text{redshift parameter}})} \qquad (1.4.4)$$

In a FLRW universe, there is a one-to-one correspondence between t, a and z $\Rightarrow$ we use them interchangeably as a measure of time.

2. The hot universe

2.1 Equilibrium thermodynamics

In general, for a gas of particles:

$$n_i(x) = \frac{g_i}{(2\pi)^3}\int d^3p\ f_i(\vec{p}, x) \qquad \text{number density}$$

$$\rho_i(x) = \frac{g_i}{(2\pi)^3}\int d^3p\ E(|\vec{p}|)\ f_i(\vec{p}, x) \qquad \text{energy density}$$

$$P_i(x) = \frac{g_i}{(2\pi)^3}\int d^3p\ \frac{|\vec{p}|^2}{3E}\ f_i(\vec{p}, x) \qquad \text{Pressure}$$

(2.1.1)

① $|\vec{p}|$ = proper momentum measured by a comoving observer
$E(|\vec{p}|) = \sqrt{|\vec{p}|^2 + m^2}$

② g_i = internal degrees of freedom
e.g., $g_\gamma = 2$ for photons (2 polarisation states)
$g_e = 4$ for electrons + positrons
(2x spin up, 2x spin down)

③ $f_i(\vec{p}, x)$ = occupancy function or phase space distribution
$= f_i(|\vec{p}|)$ if homogeneous & isotropic.

What is $f(|\vec{p}|)$?

In the early universe, a scattering process is said to be in thermal equilibrium if the particles involved scatter many times before space has time to expand significantly, i.e.,

$$\underbrace{\Gamma_{\text{interaction}}}_{\text{scattering rate per particle}} \gg \underbrace{H}_{\text{expansion rate}} \qquad (2.1.2)$$

$$\Gamma_{\text{interaction}} \sim \underbrace{\sigma}_{\text{cross-section}}\ \underbrace{n}_{\text{number density}}\ \underbrace{v}_{\text{relative speed}}$$

When a scattering process is in thermal (or kinetic) equilibrium, the particles involved have phase space distributions given by

$$f_i(|\vec{p}|) = \frac{1}{\exp\left(\frac{E-\mu_i}{T_i}\right) \pm 1} \qquad \begin{matrix} + \text{ fermion} \\ - \text{ boson} \end{matrix} \qquad (2.1.3)$$

T_i = temperature

μ_i = chemical potential.

e.g., elastic scattering processes such as

$$e^- + \gamma \leftrightarrow e^- + \gamma \qquad (2.1.4)$$

lead to equilibrium phase space distributions for γ and e^-, with $T_{e^-} = T_\gamma$.

If inelastic processes such as

$$i + j \leftrightarrow k + l \qquad \Bigg\| \quad \text{e.g., } e^-e^+ \leftrightarrow \gamma\gamma, \quad \nu_e n \leftrightarrow p e^- \qquad (2.1.5)$$

are in equilibrium, then

$$\mu_i + \mu_j = \mu_k + \mu_l \qquad (2.1.6)$$

i.e., i, j, k and l are also in chemical equilibrium.

Importantly,

$$\underbrace{\phi}_{\text{some particle}}\ \underbrace{\bar{\phi}}_{\text{its antiparticle}} \longleftrightarrow \overset{(\mu^+\mu^-,\, \tau, \bar{\tau} \ldots)}{e^+e^-} \longleftrightarrow \gamma\gamma \quad (\text{in equilibrium})$$

leads to: $\mu_\phi + \mu_{\bar{\phi}} = \cdots = 2\mu_\gamma = 0$ ‖ because the number of photons in the universe is not conserved.

$$\Rightarrow \boxed{\mu_{\bar{\phi}} = -\mu_\phi} \qquad (2.1.7)$$

μ for ϕ & $\bar{\phi}$ while in chemical equilibrium with γ

Equilibrium energy densities || assuming $|\mu_i| \ll T_i$

	Relativistic fermions, $m_i \ll T_i$	Relativistic bosons	Nonrelativistic $m_i \gg T_i$
n_i	$\frac{3}{4}\frac{\zeta(3)}{\pi^2} g_i T_i^3$	$\frac{\zeta(3)}{\pi^2} g_i T_i^3$	$g_i \left(\frac{m_i T_i}{2\pi}\right)^{3/2} \exp\left[-\frac{(m_i - \mu_i)}{T_i}\right]$
ρ_i	$\frac{7}{8}\frac{\pi^2}{30} g_i T_i^4$	$\frac{\pi^2}{30} g_i T_i^4$	$m_i n_i + \frac{3}{2} T_i n_i$
P_i	$\frac{\rho_i}{3}$	$\frac{\rho_i}{3}$	$T_i n_i \ll m_i n_i = \rho_i$

Clearly, when in equilibrium, $\rho_{\text{nonrel}} \ll \rho_{\text{rel}}$

$$\Rightarrow \sum_{\substack{i=\text{all} \\ \text{species}}} \rho_i \simeq \sum_{\substack{i=\text{relativistic} \\ \text{species}}} \rho_i \qquad (2.1.8)$$

It is then useful to write:

$$\boxed{\begin{aligned} &\rho_{\text{total}} = \sum_{\substack{i=\text{rel} \\ \text{species}}} \rho_i = \frac{\pi^2}{30} g_* T_\gamma^4 \\ &\text{where:} \\ &\qquad g_* \equiv \sum_{\substack{i=\text{rel} \\ \text{bosons}}} g_i \left(\frac{T_i}{T_\gamma}\right)^4 + \frac{7}{8} \sum_{\substack{i=\text{rel} \\ \text{fermions}}} \left(\frac{T_i}{T_\gamma}\right)^4 \end{aligned}} \qquad (2.1.9)$$

effective massless degrees of freedom

<u>Note</u>: g_* is a function of temperature and particle physics model, since at different T, different particle species satisfy $m_i \ll T_i$

2.2 Entropy

Thermal equilibrium = a maximal entropy state. If equilibrium is maintained, then the entropy in a comoving volume element stays constant.

Define the entropy density $s(T)$ as the entropy per unit volume. Then:

equilibrium $\Rightarrow$
$$\boxed{\begin{aligned} s(T) &\propto a^{-3} \\ &= \frac{\rho(T) + P(T)}{T} \end{aligned}}$$

(2.2.1)

assuming $\mu \ll T$

from 2nd law of thermodynamics

$T(ds(T)V) = d(\rho(T)V) + P(T)\,dV$

Like ρ_{total}, s_{total} is dominated by relativistic species. Thus, analogous to (2.1.9), we write:

$$\boxed{\begin{aligned} s_{total} &= \frac{2\pi^2}{45} g_{*s} T_\gamma^3 \\ \text{where: } g_{*s} &\equiv \sum_{i=\text{rel bosons}} g_i \left(\frac{T_i}{T_\gamma}\right)^3 + \frac{7}{8} \sum_{i=\text{rel fermions}} g_i \left(\frac{T_i}{T_\gamma}\right)^3 \end{aligned}}$$

(2.2.2)

If all T_i's are the same, then $g_{*s} = g_*$ of (2.1.9).

Compare (2.2.1) and (2.2.2):

$$s \propto a^{-3}$$
$$\propto g_{*s} T_\gamma^3$$

$$\Rightarrow \boxed{T_\gamma \propto \underbrace{g_{*s}^{-1/3}}_{\text{Generally temperature-dependent}} a^{-1}}$$

Evolution of the photon temperature

(2.2.3)

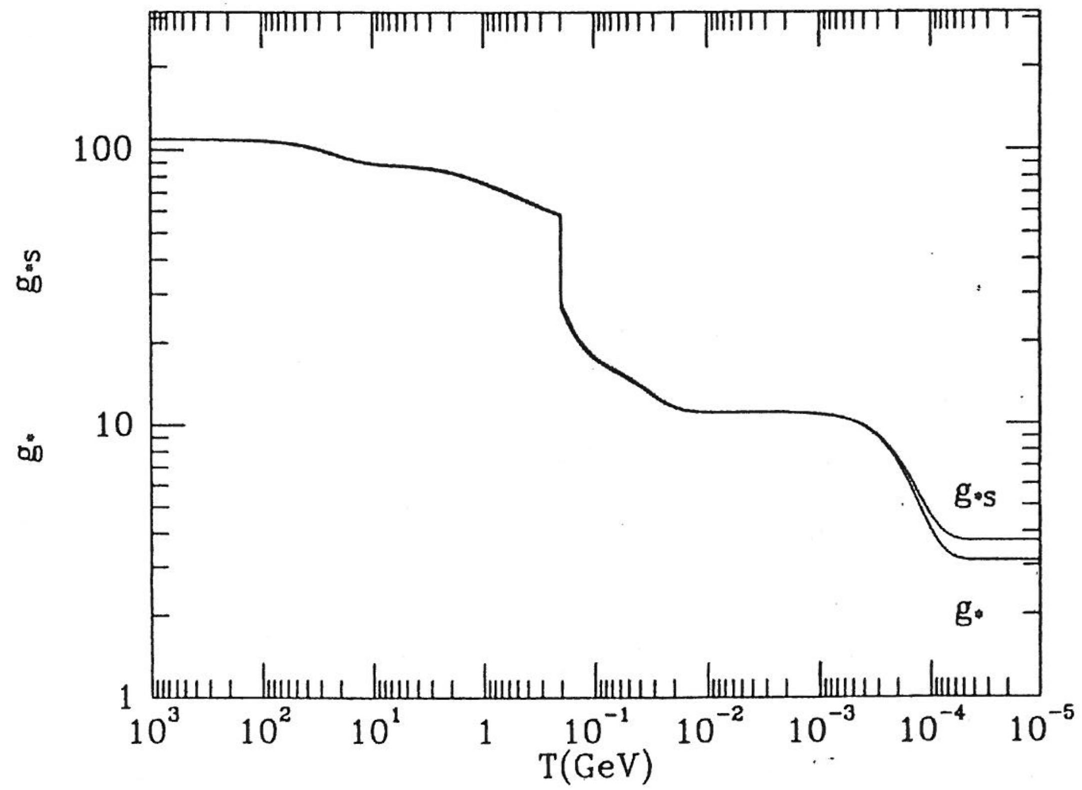


Fig. 3.5: The evolution of $g_*(T)$ as a function of temperature in the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ theory.

2.3 Decoupling

As the universe cools, $\Gamma_{\text{interaction}}$ decreases. When

$$\Gamma_{\text{interaction}} \ll H \qquad (2.3.1)$$

interactions are too slow to keep up with expansion
$\Rightarrow$ decoupling.

Example: Neutrino decoupling || $\nu e \leftrightarrow \nu e$, $e^+e^- \leftrightarrow \nu\bar{\nu}$ etc.

$$\Gamma_{\text{interaction}} \sim G_F^2 T^5 \quad \Big\| \quad \sigma \sim G_F^2 E^2, \; n \sim T^3$$

$$H \sim \frac{T^2}{M_{pl}} \quad \Big\| \quad \text{from Friedmann eqn} \quad M_{pl}^2 \equiv \frac{1}{G} \qquad (2.3.2)$$

Then, $\Gamma_{\text{int}} \ll H$ implies:

$$\boxed{T \ll \left[G_F^2 M_{pl}\right]^{-1/3} \sim O(1)\,\text{MeV}} \qquad (2.3.3)$$

Neutrino decoupling temperature

Evolution of $f(|\vec{p}|)$ after decoupling

In general, we need to solve the Boltzmann equation to find $f(|\vec{p}|)$ after decoupling (more later). But there are two special cases:

① Decoupling when particle species is highly relativistic (e.g., neutrinos). Then

$$f(|\vec{p}|) \simeq \frac{1}{\exp\left(\frac{|\vec{p}|-\mu}{T}\right) \pm 1} \qquad (2.3.4)$$

with $T \propto a^{-1}$ and $\mu \propto a^{-1}$, i.e., particle species maintains its relativistic FD or BE equilibrium distribution.

② Decoupling when particle species is very non-relativistic. Then,

$$f(|\vec{p}|) \simeq \exp\left[-\frac{(m-\mu)}{T}\right] \exp\left(\frac{-|\vec{p}|^2}{2mT}\right) \qquad (2.3.5)$$

with $T \propto a^{-2}$ and $\mu = m + [\mu(T_0) - m)] \frac{T}{T_0}$.

__Note:__ (2.3.5) is the non-relativistic ($m \gg T$) limit of

$$f(|\vec{p}|) = \frac{1}{\exp\left(\frac{E-\mu}{T}\right) \pm 1}$$

$$\simeq \frac{1}{\exp\left[\left(m + \frac{|\vec{p}|^2}{2m} - \mu\right)/T\right] \pm 1}$$

$$= \frac{\exp\left[-\frac{(m-\mu)}{T}\right]}{\exp\left(\frac{|\vec{p}|^2}{2mT}\right) \pm \underbrace{\exp\left[-\frac{(m-\mu)}{T}\right]}_{\ll 1\ ;\ m \gg \mu.}} \qquad (2.3.6)$$

Therefore, particle species that decouple while non-relativistic maintain their Maxwell-Boltzmann equilibrium distribution.

2.4 Neutrino temperature today

	events	Relativistic species	temperature	Entropy
100MeV		γ, ν, e	$T_\gamma = T_e = T_\nu$ (thermal equilibrium)	$S = S_e + S_\gamma + S_\nu$ Conserved together
1MeV	Neutrino decoupling	γ, ν, e	$T_\gamma = T_e \propto g_{*s}^{-1/3} a^{-1}$ $T_\nu \propto a^{-1} = T_\gamma$ (since g_{*s} is constant)	$S_{e\gamma}(a_1) = S_e + S_\gamma$ $S_\nu(a_1)$ conserved separately
0.3MeV	e^+e^- become nonrelativistic $e^+e^- \to \gamma\gamma$ (since $m_e \gg T_\gamma$)	γ, ν	$T_\gamma = T_e$ $T_\nu = ?$	$S_{e\gamma}(a_2) = S_e + S_\gamma$ $S_\nu(a_2)$

t

Conservation of entropy in the $e\gamma$ system:

$$S_{e\gamma}(a_1) a_1^3 = S_{e\gamma}(a_2) a_2^3$$

where

$$S_{e\gamma}(a_1) \propto \left(\overset{2}{g_\gamma} + \frac{7}{8} \overset{4}{g_e}\right) T_\gamma^3(a_1) = \frac{11}{2} T_\gamma^3(a_1)$$

$$S_{e\gamma}(a_2) \propto g_\gamma T_\gamma^3(a_2) = 2 T_\gamma^3(a_2)$$

$$\Rightarrow \quad \frac{T_\gamma(a_1)}{T_\gamma(a_2)} = \left(\frac{4}{11}\right)^{1/3} \frac{a_2}{a_1} \qquad (2.4.1)$$

Conservation of entropy in ν:

$$S_\nu(a_1) a_1^3 = S_\nu(a_2) a_2^3$$

$$\Rightarrow \quad \frac{a_2}{a_1} = \frac{T_\nu(a_1)}{T_\nu(a_2)} \qquad (2.4.2)$$

Combining (2.4.1) and (2.4.2) gives

$$\frac{T_\gamma(a_1)}{T_\gamma(a_2)} = \left(\frac{4}{11}\right)^{1/3} \frac{T_\nu(a_1)}{T_\nu(a_2)} \qquad (2.4.3)$$

But $T_\gamma(a_1) = T_\nu(a_1)$ (before e^+e^- annihilation).
Thus:

$$\boxed{T_\nu(a_2) = \left(\frac{4}{11}\right)^{1/3} T_\gamma(a_2)} \qquad (2.4.4)$$

Given $T_\gamma(\text{today}) = 2.73\,K$ (CMB temperature),

$$\Rightarrow T_\nu(\text{today}) = 1.95\,K \qquad (2.4.5)$$

$\Rightarrow$ Cosmic neutrino background

Energy density of the cosmic neutrino background

① If massless:

$$\rho_\nu(t_0) = 3 \times \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \rho_\gamma(t_0) \qquad (2.4.6)$$

$$\Rightarrow \Omega_\nu \equiv \frac{\rho_\nu(t_0)}{\rho_{crit}(t_0)} = 1.68 \times 10^{-5} h^{-2}$$

$H_0 = 100\,h\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$

② If massive and $m_\nu \gg T_\nu(t_0)$:

$$n_{\nu_i}(t_0) = \frac{3}{11} n_\gamma(t_0)$$

One neutrino flavour

$$\Rightarrow \Omega_\nu \equiv \frac{\rho_\nu(t_0)}{\rho_{crit}(t_0)} = \frac{\sum_i m_i n_{\nu_i}(t_0)}{\rho_{crit}(t_0)}$$

$$= \frac{\sum_i m_{\nu_i}}{94 h^2 \mathrm{eV}} \qquad (2.4.7)$$

2.5 Boltzmann equation

The evolution of the phase space distribution $f_i(\vec{p}, x)$ can be tracked using the Boltzmann equation:

$$\boxed{\underbrace{\tilde{p}^\alpha \frac{\partial f}{\partial x^\alpha} - \Gamma^i_{\alpha\beta} \tilde{p}^\alpha \tilde{p}^\beta \frac{\partial f}{\partial \tilde{p}^i}} = C[f]} \qquad \begin{matrix} \alpha, \beta = 0,1,2,3 \\ i = 1,2,3 \end{matrix} \qquad (2.5.1)$$

$L[f]$ = Liouville operator (gravitational effects)

$C[f]$ = Collision operator (non-gravitational effects)

In a FLRW universe, (2.5.1) becomes:

$$\boxed{\frac{\partial f}{\partial t} - \frac{\dot{a}}{a} |\vec{p}| \frac{\partial f}{\partial |\vec{p}|} = \frac{1}{E} C[f]} \qquad (2.5.2)$$

(2.5.2) describes the full phase space evolution. But we are usually more interested in bulk quantities such as the number density of a particle species. Therefore we integrate (2.5.2) over momentum:

$$\frac{g}{(2\pi)^3} \int d^3p \left[\frac{\partial f}{\partial t} - \frac{\dot{a}}{a} |\vec{p}| \frac{\partial f}{\partial |\vec{p}|} \right] = \frac{g}{(2\pi)^3} \int d^3p \, \frac{1}{E} C[f]$$

$$\Rightarrow \quad \boxed{\frac{dn}{dt} + 3 \frac{\dot{a}}{a} n = \frac{g}{(2\pi)^3} \int \frac{d^3p}{E} C[f]} \qquad (2.5.3)$$

If collisionless, i.e., $C[f] = 0$, then $n \propto a^{-3}$, as expected.

Collision term : for a process $i+j \leftrightarrow k+l$

$$\frac{g_i}{(2\pi)^3}\int d^3p \, \frac{C[f_i]}{E_i} = -\int d\Pi_i \, d\Pi_j \, d\Pi_k \, d\Pi_l$$

$$\times (2\pi)^4 \delta_D^{(4)}(\tilde{P}_i + \tilde{P}_j - \tilde{P}_k - \tilde{P}_l)$$

$$\times \Big[|\mathcal{M}|^2_{i+j\to k+l} f_i f_j (1 \pm f_k)(1 \pm f_l)$$

$$- |\mathcal{M}|^2_{k+l\to i+j} f_k f_l (1 \pm f_i)(1 \pm f_j) \Big]$$

where : (2.5.4)

$$d\Pi_i \equiv \frac{g_i}{(2\pi)^3}\frac{d^3p_i}{2E_i} \quad \text{for species } i$$

$|\mathcal{M}|^2$ = matrix element of interaction (averaged over initial & final spins)

$\delta_D^{(4)}(\tilde{P}_i + \tilde{P}_j - \tilde{P}_k - \tilde{P}_l)$ = Dirac delta (Energy-momentum conservation)

4-momentum

$$(1 \pm f_i) = \begin{cases} + \text{ Bose enhancement} & \text{if } i = \text{boson} \\ - \text{ Pauli blocking} & \text{if } i = \text{fermion} \end{cases}$$

If scattering process obeys CP or T invariance,

$$|\mathcal{M}|^2_{i+j\to k+l} = |\mathcal{M}|^2_{k+l\to i+j} \equiv |\mathcal{M}|^2 \qquad (2.5.5)$$

Useful approximations

① Ignore quantum statistics, i.e.,

$$\text{set } 1 \pm f_i \to 1$$

and

$$f_i^{eq} = \exp\left(-\frac{E_i + \mu_i}{T_i}\right) \quad \Bigg\| \text{ Maxwell-Boltzmann statistics} \qquad (2.5.6)$$

② In general, elastic scattering processes remain in equilibrium down to a lower temperature than inelastic ones, e.g., $\gamma e^- \leftrightarrow \gamma e^-$ versus $e^+ e^- \leftrightarrow \gamma\gamma$.

$\Rightarrow$ kinetic equilibrium can be assumed, i.e.,

$$f_i = \exp\left(\frac{-E_i + \mu_i}{T_i}\right)$$

$$\text{and} \qquad T_i = T_j = T_k = T_l \equiv T \qquad (2.5.7)$$

although in general $\mu_i + \mu_j \neq \mu_k + \mu_l$ (chemical equilibrium does not necessarily hold).

Define a <u>reference</u> equilibrium density:

$$n_i^{(0)} \equiv g_i \int \frac{d^3p}{(2\pi)^3} \, e^{-E_i/T} \qquad (2.5.8)$$

Then:

$$\frac{n_i}{n_i^{(0)}} = \frac{g_i \int \frac{d^3p}{(2\pi)^3} \, e^{-\frac{E_i + \mu_i}{T}}}{g_i \int \frac{d^3p}{(2\pi)^3} \, e^{-E_i/T}} = e^{\mu_i/T}$$

$$(2.5.7) \Rightarrow \boxed{f_i = \frac{n_i}{n_i^{(0)}} = e^{-E_i/T}} \qquad (2.5.9)$$

Then, ① + ② give:

Collision integral $= -\int d\Pi_i d\Pi_j d\Pi_k d\Pi_l \, (2\pi)^4 \delta_D^{(4)}(\tilde{P}_i + \tilde{P}_j - \tilde{P}_k - \tilde{P}_l)$

$$\times |\mathcal{M}|^2 \left[\frac{n_i}{n_i^{(0)}} \frac{n_j}{n_j^{(0)}} e^{\frac{-E_i - E_j}{T}} - \frac{n_k}{n_k^{(0)}} \frac{n_l}{n_l^{(0)}} e^{\frac{-E_k - E_l}{T}} \right]$$

assuming CP invariance

= because of energy conservation.

$$\Rightarrow \boxed{\frac{dn_i}{dt} + 3\frac{\dot{a}}{a} n_i = -n_i^{(0)} n_j^{(0)} \langle \sigma v \rangle \left[\frac{n_i}{n_i^{(0)}} \frac{n_j}{n_j^{(0)}} - \frac{n_k}{n_k^{(0)}} \frac{n_l}{n_l^{(0)}} \right]}$$

where:

$$\langle \sigma v \rangle \equiv \frac{1}{n_i^{(0)} n_j^{(0)}} \int d\Pi_i d\Pi_j d\Pi_k d\Pi_l \, e^{\frac{-E_i - E_j}{T}} \times (2\pi)^4 \delta_D^{(4)}(\tilde{P}_i + \tilde{P}_j - \tilde{P}_k - \tilde{P}_l) |\mathcal{M}|^2$$

(2.5.10)

Thermally averaged cross-section

2.6 Applications of the Boltzmann equation

Freeze-out of WIMPs

WIMP = weakly interacting particles, a candidate for cold dark matter.

Consider the process: $\psi\bar{\psi} \leftrightarrow X\bar{X}$

WIMP (ψ) ; X ↳ some standard model particle, e.g., e, ν ..

The Boltzmann equation for n_ψ is:

$$\frac{dn_\psi}{dt} + 3\frac{\dot{a}}{a}n_\psi = -n_\psi^{(0)} n_{\bar{\psi}}^{(0)} \langle \sigma_{\psi\bar{\psi}\leftrightarrow X\bar{X}} v \rangle \left[\frac{n_\psi n_{\bar{\psi}}}{n_\psi^{(0)} n_{\bar{\psi}}^{(0)}} - \overbrace{\frac{n_X n_{\bar{X}}}{n_X^{(0)} n_{\bar{X}}^{(0)}}}^{\to 1}\right] \quad (2.6.1)$$

Two more approximations:

① X and $\bar{X}$ have other interactions to keep them in kinetic equilibrium $\Rightarrow$ $n_X \cong n_X^{(0)}$, $n_{\bar{X}} \cong n_{\bar{X}}^{(0)}$

② $n_\psi = n_{\bar{\psi}}$

Then (2.6.1) becomes

$$\boxed{\frac{dn_\psi}{dt} + 3\frac{\dot{a}}{a}n_\psi = -\sum_{\substack{\text{all}\\ \text{annihilation}\\ \text{channels}}} \langle \sigma_{\psi\bar{\psi}\leftrightarrow X\bar{X}} v \rangle \left[n_\psi^2 - n_\psi^{(0)2}\right]} \quad (2.6.2)$$

Alternatively:

$$\frac{x}{Y_\psi^{(0)}}\frac{dY_\psi}{dx} = \frac{\Gamma_A}{H}\left[1 - \left(\frac{Y_\psi}{Y_\psi^{(0)}}\right)^2\right] \quad (2.6.3)$$

where $Y_\psi \equiv \frac{n_\psi}{s}$; $x = \frac{m_\psi}{T}$

$\Gamma_A \equiv n_\psi \langle \sigma v \rangle$

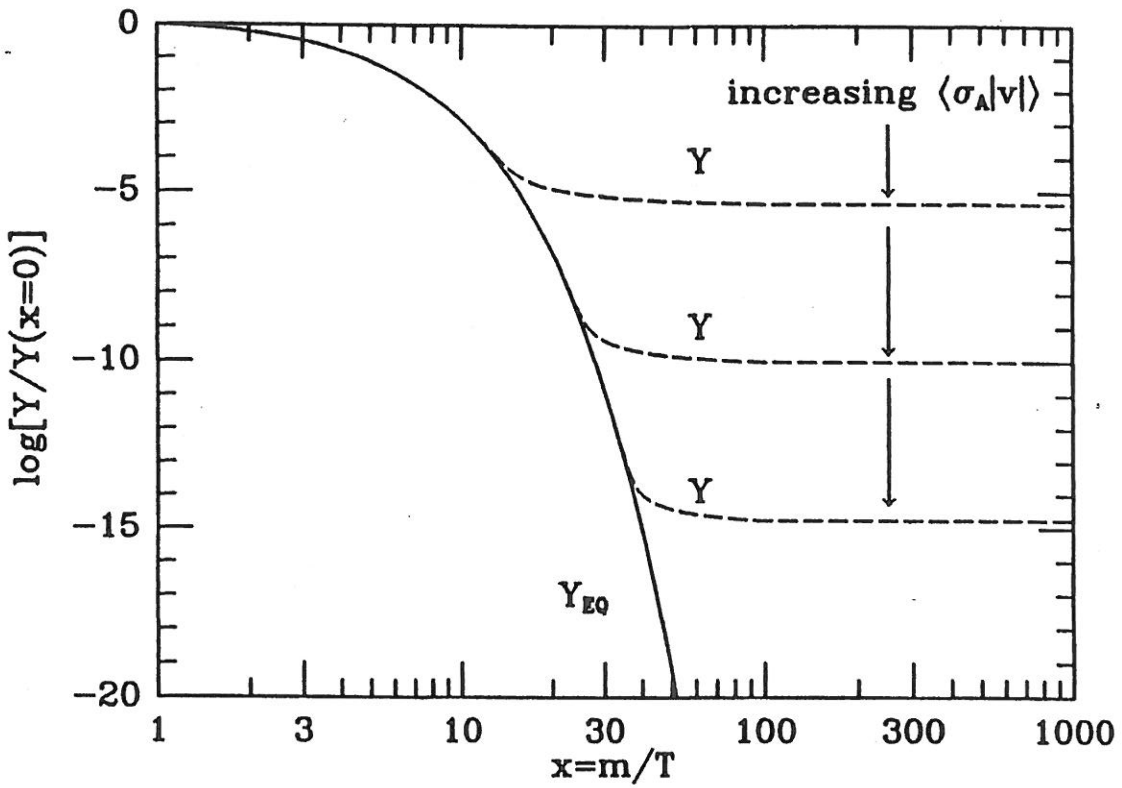


Fig. 5.1: The freeze out of a massive particle species. The dashed line is the actual abundance, and the solid line is the equilibrium abundance.

Recombination and photon decoupling

When $T > O(1)\,\text{eV}$, Thomson scattering keeps γ and e^- in equilibrium:

$$\gamma e^- \leftrightarrow \gamma e^- \qquad (2.6.4)$$

free electron (not in an atom)

But the free electron density, n_e, is governed by

$$e + p \leftrightarrow H + \gamma \qquad \text{recombination} \qquad (2.6.5)$$

neutral hydrogen

We track n_e using:

$$\frac{dn_e}{dt} + 3\frac{\dot{a}}{a} n_e = n_e^{(0)} n_p^{(0)} \langle \sigma_T v \rangle \left[\frac{n_H}{n_H^{(0)}} - \frac{n_e^2}{n_e^{(0)} n_p^{(0)}} \right] \qquad (2.6.6)$$

When n_e becomes so low that the process (2.6.4) falls out of equilibrium $\Rightarrow$ photons free-stream $\Rightarrow$ Cosmic microwave background.

Baryogenesis via leptogenesis

CP violating, out-of-equilibrium decay

$$N \to \Phi\, \ell \qquad (2.6.7)$$

heavy RH Majorana neutrino; Higgs; lepton

CP violation means

Tree level matrix element; CP violation from 1-loop

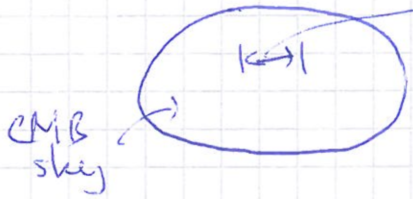
$$|\mathcal{M}_{N\to\Phi\ell}|^2 = |\mathcal{M}_{\bar\Phi\bar\ell\to N}|^2 = |\mathcal{M}_0|^2(1+\varepsilon)$$
$$|\mathcal{M}_{N\to\bar\Phi\bar\ell}|^2 = |\mathcal{M}_{\Phi\ell\to N}|^2 = |\mathcal{M}_0|^2(1-\varepsilon) \qquad (2.6.8)$$

Use BE to track n_N and the lepton asymmetry $L \equiv \dfrac{n_\ell - n_{\bar\ell}}{n_\gamma}$.

3. Inflation

3.1 Motivation

① The horizon problem: The CMB is remarkably uniform despite the fact that it is made up of many causally disconnected regions.



Causal region:

$$\Delta\theta \sim \frac{d_H}{d_A} \sim \frac{\mathcal{O}(0.1)\,\mathrm{Mpc}}{\mathcal{O}(10)\,\mathrm{Mpc}} \sim 1^\circ \qquad (3.1.1)$$

physical horizon at γ decoupling

angular diameter distance to the last scattering surface

numbers from standard cosmology.

② The flatness problem: The universe appears to have a flat geometry today, $|\Omega_k| \leq 0.01$, from observations. But, from the Friedmann equation:

$$\frac{d}{dt}\left[\Omega(t) - 1\right] = -2\ddot{a}\left[\Omega(t) - 1\right] \qquad \Bigg\| \quad \Omega(t) \equiv \frac{\rho(t)}{\rho_{crit}(t)} \qquad (3.1.2)$$

$\ddot{a} < 0$ during MD and RD

i.e., $\Omega(t) - 1 = 0$ is an unstable fixed point. In order to satisfy:

$$|\Omega(t_0) - 1| = |\Omega_k| \lesssim 0.01 \qquad (3.1.3)$$

we must have:

time when $T \sim M_{pl} \sim 10^{19}\,\mathrm{GeV}$

$$|\Omega(t_{pl}) - 1| \leq 10^{-60} \qquad (3.1.4)$$

$\Rightarrow$ Too much fine-tuning is required to make the universe appear flat today.

③ The monopole problem: GUTs predict topological defects (monopoles, cosmic strings, domain walls, etc.). Generally, we expect one defect per causally connected region at the time of creation ($T > 10^{16}$ GeV)
⇒ there must be many of these objects in the observable universe today. Where are they?.

Main idea of inflation

Introduce a phase of accelerated expansion $\ddot{a} > 0$ before radiation domination.

① Horizon problem can be solved because the physical horizon at t_{dec} can be made much larger than the estimate (3.1.1).

② Flatness problem solved because a positive $\ddot{a}$ will drive any initial $\Omega(t) - 1$ to zero given sufficient time.

③ Rapid expansion dilutes abundance of topological defects, provided inflation occurs after production of defects.

3.2 Scalar field dynamics

Lagrangian density for a scalar field φ : $\left(\begin{matrix}\text{minimally}\\ \text{coupled}\end{matrix}\right)$

$$\mathcal{L}_\varphi = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi) \qquad \Big\|\ \varphi = \text{inflaton} \qquad (3.2\text{-}1)$$

minus sign because our metric has $(-+++)$

$$= -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)$$

Consider a homogeneous φ (i.e., $\partial_i\varphi = 0$), then from

$$T_{\mu\nu} = \frac{-2}{\sqrt{g}}\frac{\partial}{\partial g^{\mu\nu}}\left(\sqrt{g}\,\mathcal{L}_m\right) \qquad (3.2\text{-}2)$$

and assuming the FLRW metric (1.1.1), we find:

$$\begin{aligned} T^0{}_0 &= -\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - V(\phi) \quad ; \quad T^0{}_i = 0 \\ \underset{i=j}{T^i{}_j} &= \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - V(\phi) \quad ; \quad \underset{i\neq j}{T^i{}_j} = 0 \end{aligned} \qquad (3.2\text{-}3)$$

Compare with a perfect fluid:

$$T^\mu{}_\nu = \mathrm{diag}(-\rho, P, P, P)$$

$$\Rightarrow \boxed{\begin{aligned} \rho_\phi &= \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + V(\phi) \\ P_\phi &= \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - V(\phi) \end{aligned}} \qquad (3.2.4)$$

energy density and pressure of a homogeneous scalar field

If $V(\phi) \gg \dot{\phi}$:

$$w_\phi = \frac{P_\phi}{\rho_\phi} \simeq -1 \qquad (3.2.5)$$

$\Rightarrow$ accelerated expansion if φ dominates the energy density.

Evolution equation for φ :

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{\partial V}{\partial\phi} = 0 \qquad \Big\|\ \text{from } \nabla^\mu T_{\mu\nu} = 0 \text{ or } \frac{d\rho}{dt} + 3H(\rho + P) = 0 \qquad (3.2.6)$$

3.3 Basic picture of inflation

Consider a causally connected patch of the universe at some early time.

① If the energy density in the patch is dominated by $V(\phi)$ of a slowly varying ϕ ("slow-roll"), i.e.,

$$\frac{1}{2}\dot{\phi}^2 \ll V(\phi) \qquad \| \; KE \ll PE \qquad (3.3.1)$$

or equivalently: $$\ddot{\phi} \ll 3H\dot{\phi} \simeq -V_{,\phi} \qquad \| \; V_{,\phi} \equiv \frac{\partial V}{\partial \phi} \qquad (3.3.2)$$

then $w_\phi \equiv \frac{P_\phi}{\rho_\phi} \simeq -1 \Rightarrow$ inflationary phase.

(3.3.1) and (3.3.2) can be translated into constraints on the form of $V(\phi)$:

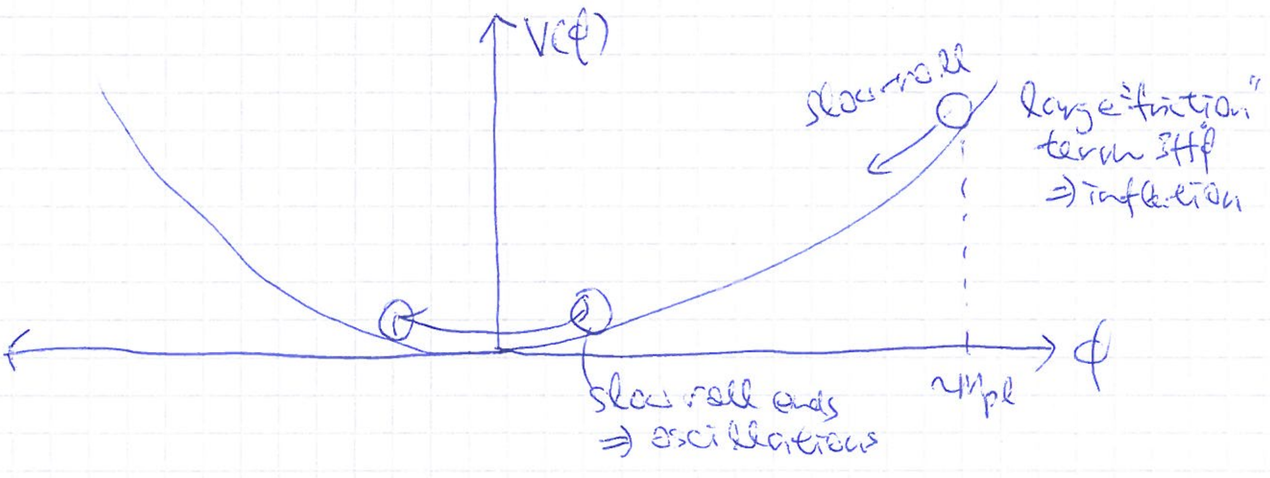
slow roll parameters

$$\left\{ \begin{array}{l} \varepsilon \equiv \frac{m_{pl}^2}{16\pi}\left(\frac{V_{,\phi}}{V}\right)^2 = -\frac{\dot{H}}{H^2} \overset{\text{inflation}}{\simeq} \frac{3}{2}\frac{\dot{\phi}^2}{V} \\ \eta \equiv \frac{m_{pl}^2}{8\pi}\left(\frac{V_{,\phi\phi}}{V}\right) \\ \vdots \end{array} \right. \qquad (3.3.3)$$

During inflation, $\varepsilon, \eta \ldots \ll 1$

② When $\varepsilon, \eta \to 1$, inflation ends.

Example: $V(\phi) = \frac{1}{2}m^2\phi^2$ (large-field models)



After inflation: Reheating

Reheating = conversion of energy in ϕ into relativistic (standard model) particles

$\Rightarrow$ universe enters into radiation domination.

The actual process of reheating is not well understood. But suppose:

① It is a simple ϕ decay process, with decay width Γ_ϕ.

② All energy in ϕ is converted instantaneously into radiation when $H \sim \Gamma_\phi$.

Then:

$$\Gamma_\phi \sim H = \sqrt{\frac{8\pi G}{3}\rho_R} = \sqrt{\frac{8\pi}{3m_{pl}^2}\left(\frac{\pi^2}{30}\right) g_* T_{RH}^4}$$

Friedmann eqn

radiation

$$\Rightarrow \boxed{\begin{aligned} T_{RH} &= \left(\frac{90}{8\pi^3 g_*}\right)^{1/4}\sqrt{\Gamma_\phi m_{pl}} \\ &= 0.2\left(\frac{200}{g_*}\right)^{1/4}\sqrt{\Gamma_\phi m_{pl}} \end{aligned}}$$

Reheating temperature (3.3.4)

4. Structure formation

4.1 Overview

The formation of gravitationally bound objects such as dark matter halos, galaxies, clusters, etc. proceeds through gravitational instability

The various stages of structure formation:

① Quantum fluctuations of the inflaton field induce perturbations in the spacetime metric during inflation.

② Linear evolution of perturbations
⇒ CMB anisotropies
⇒ Large-scale distribution of matter (> 10 Mpc)

③ Nonlinear evolution on small scales (< 10 Mpc) at late times.
⇒ formation of bound structures
⇒ halo density profiles

numerical simulations

t

4.2 Describing small perturbations

We use cosmological perturbation theory. Define the spacetime metric:

$$g_{\mu\nu} = \underbrace{\bar{g}_{\mu\nu}}_{\text{FLRW}} + \underbrace{a^2 h_{\mu\nu}}_{\text{perturbation}} \qquad |h_{\mu\nu}| \ll 1 \qquad (4.2.1)$$

and the stress-energy tensor:

$$T_{\mu\nu} = \underbrace{\bar{T}_{\mu\nu}}_{=\mathrm{diag}(\bar{\rho},\, g_{\mu\nu}\bar{P})} + \delta T_{\mu\nu} \qquad (4.2.2)$$

(homogeneous & isotropic)

We use the Einstein equation to relate $h_{\mu\nu}$ and $\delta T_{\mu\nu}$.

Note that in general:

$h_{\mu\nu}$ = 10 d.o.f.
- → 6 physical d.o.f
  - → 2 scalars, 2x 1d.o.f
  - → 1 vector; 1x2 d.o.f
  - 1 tensor; 1x2 d.o.f (gravitational waves)
- → 4 gauge modes (related to our choice of coordinate system)

To linear order in the small quantities, only the scalar degrees of freedom in $h_{\mu\nu}$ are relevant for structure formation. One way to parametrise these scalar degrees of freedom is:

$$ds^2 = a^2 \left\{ -[1 + 2\Psi(x,\tau)]\, d\tau^2 + [1 - 2\Phi(x,\tau)]\, \gamma_{ij} dx^i dx^j \right\} \qquad (4.2.3)$$

(Conformal Newtonian gauge)

↳ $\underbrace{d\tau}_{\text{conformal time}} = \frac{dt}{a}$ ← cosmic time

In the same gauge, the stress-energy tensor (to 1st order in small quantities) is:

$$T^\mu{}_\nu = \begin{pmatrix} -\bar\rho & & & \\ & \bar P & & \\ & & \bar P & \\ & & & \bar P \end{pmatrix} + \underbrace{\left(\begin{array}{c|ccc} -\delta\rho & (\bar\rho+\bar P)\cdot v_i & & \\ \hline -(\bar\rho+\bar P)v_i & \delta P & & \\ & & \delta P & \\ & & & \delta P \end{array}\right)}_{\text{Perfect fluid}} + \underbrace{\left(\begin{array}{c|c} & \\ \hline & \Pi^i{}_j \end{array}\right)}_{\text{additional term for non-perfect fluid}} \quad (4.24)$$

$\Pi^i{}_j$ = anisotropic stress.

General expressions relating Ψ, Φ to $\delta\rho$, δP, etc. can be found in, e.g., Ma & Bertschinger astro-ph/9506072. Here, we only look at a simple example.

4.3 Example: Evolution of CDM density perturbations during matter domination

Suppose the universe's energy density is dominated by cold dark matter (CDM).

CDM means $\underbrace{\bar{P},\ \delta P,\ \Pi^i{}_j = 0}_{\text{i.e., no velocity dispersion.}}$ by definition

Then:

$$T^\mu{}_\nu = \underbrace{\left(\begin{array}{c|c} -\bar{\rho} & 0 \\ \hline 0 & 0 \end{array}\right)}_{\bar{T}^\mu{}_\nu} + \underbrace{\left(\begin{array}{c|c} -\bar{\rho}\delta & \bar{\rho} v_i \\ \hline -\bar{\rho} v^i & 0 \end{array}\right)}_{\delta T^\mu{}_\nu} \qquad (4.3.1)$$

Using the Einstein equation to relate Ψ, Φ to $\delta T^\mu{}_\nu$, we find:

$$\Phi = \Psi \qquad \text{Consequence of } \Pi^i{}_j = 0 \qquad (4.3.2)$$

and:

$$\boxed{k^2 \Psi = -4\pi G a^2 \bar{\rho}\left[\delta + \frac{3\mathcal{H}}{k} v^{(s)}\right] = -\frac{3}{2}\mathcal{H}^2\left[\delta + \frac{3\mathcal{H}}{k} v^{(s)}\right]} \qquad (4.3.3)$$

Notes

① We are working in Fourier space, $\vec{x} \to \vec{k}$.

② Flat spatial geometry has been assumed.

③ $\mathcal{H} = aH$ ← Hubble parameter ; $\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\tau}$.

④ $v^{(s)} = v_i k^i / k$, i.e., component of v_i parallel to the wave vector k_i.

The evolution of $\delta T^\mu{}_\nu$ is given by $\nabla_\mu T^\mu{}_\nu = 0$ (local conservation of energy-momentum).

$$\Rightarrow \boxed{\begin{aligned} &\frac{\partial \delta}{\partial \tau} + k v^{(s)} - 3\frac{\partial \Phi}{\partial \tau} = 0 \quad &\text{Continuity eq}^n \\ &\frac{\partial v^{(s)}}{\partial \tau} + \mathcal{H} v^{(s)} - k\Phi = 0 \quad &\text{Euler eq}^n \end{aligned}} \qquad (4.3.4)$$

To find the solution, we combine (4.3.3) and (4.3.4) into a 2nd order DE for Φ:

$$\left[3 + \frac{2k^2}{3\mathcal{H}^2}\right]\frac{\partial^2 \Phi}{\partial \tau^2} + \left\{\frac{2}{3\mathcal{H}}\left[\frac{9\mathcal{H}^2}{2} + 2k^2\right] + (9\mathcal{H}^2 + k^2)\left(3 + \frac{2k^2}{3\mathcal{H}^2}\right)\left(\frac{9\mathcal{H}^2}{2} + k^2\right)\right\}\frac{\partial \Phi}{\partial \tau} = 0 \qquad (4.3.5)$$

(4.3.5) is in the form $\alpha \frac{\partial^2 \Phi}{\partial \tau^2} + \beta \frac{\partial \Phi}{\partial \tau} = 0 \Rightarrow \Phi = \text{constant}$ is a solution. (There is also a second solution which decays with τ.)

$$\Rightarrow \boxed{\Phi = \text{constant in a CDM dominated universe}}$$

For perturbations with wavelengths smaller than the Hubble radius, i.e., $k \gg \mathcal{H}$,

$$(4.3.3) \Rightarrow \underbrace{k^2 \Phi}_{\text{constant}} \simeq -4\pi G a^2 \underbrace{\bar{\rho}}_{\propto a^{-3}} \delta \qquad (4.3.6)$$

Thus: $$\boxed{\delta \propto a} \qquad (4.3.7)$$

CDM density perturbations grow like the scale factor inside the Hubble horizon during matter domination.

4.4 Hot/warm dark matter

HDM or WDM has velocity dispersion ⇒ larger δP and Π^i_j have important consequences for structure formation.

A full quantitative discussion is quite complex (see, e.g., Ma & Bertschinger, astro-ph/9506072). Here we consider the following heuristic argument for neutrino hot dark matter.

Recall that after decoupling, the homogeneous neutrino phase space distribution is

$$f(|\vec{p}|, T_\nu) = \frac{1}{\exp\left(\frac{|\vec{p}|}{T_\nu}\right) + 1} \qquad \text{Relativistic Fermi-Dirac} \qquad (4.4.1)$$

The characteristic thermal speed is then:

$$v_{th} = \frac{T_\nu}{m_\nu} = 50.4\, a^{-1} \left(\frac{eV}{m_\nu}\right) km\, s^{-1} \qquad \text{using } T_{\nu 0} = 1.95K \qquad (4.4.2)$$

This thermal motion leads to large pressure and shear stress in the neutrino gas, which counter the effect of gravity ⇒ the neutrino gas does not collapse gravitationally because particles fly away.

Define the gravitational collapse timescale:

$$\Delta t_{grav} \equiv (4\pi G \bar{\rho} a^2)^{-1/2} \qquad (4.4.3)$$

Then:

$$\boxed{\begin{aligned}\lambda_{fs} &\equiv v_{th}\,\Delta t_{grav}\\ &= 0.41\,\Omega^{-1/2}\,a^{-1/2}\left(\frac{eV}{m_\nu}\right) h^{-1}\,Mpc\end{aligned}} \qquad (4.4.4)$$

<u>Free-streaming scale</u>

i.e., at any time, density perturbations in the neutrino gas with wavelength less than λ_{fs} <u>do not</u> grow.

Now, let $a = a_{nr}$, where a_{nr} = the scale factor at which neutrinos become nonrelativistic, i.e., $m_\nu \gtrsim T_\nu(a_{nr})$.

$$\Rightarrow a_{nr} \simeq \frac{T_{\nu,0}}{m_\nu} \qquad T_{\nu,0} = 1.95\,K \sim 10^{-4}\,eV. \qquad (4.4.5)$$

Then, the <u>maximum</u> free-streaming scale is:

$$\boxed{\lambda_{fs,max} = 31.8\,\Omega^{-1/2}\left(\frac{eV}{m_\nu}\right)^{1/2} h^{-1}\,Mpc} \qquad (4.4.6)$$

i.e., unless density perturbations are regenerated by other means, primordial density perturbations in neutrinos with wavelength less than $\lambda_{fs,max}$ will be erased.

Thus, if neutrinos were the only dark matter, we would have trouble forming galaxies ($\lesssim$10kpc) and galaxy clusters ($\sim$1Mpc)!

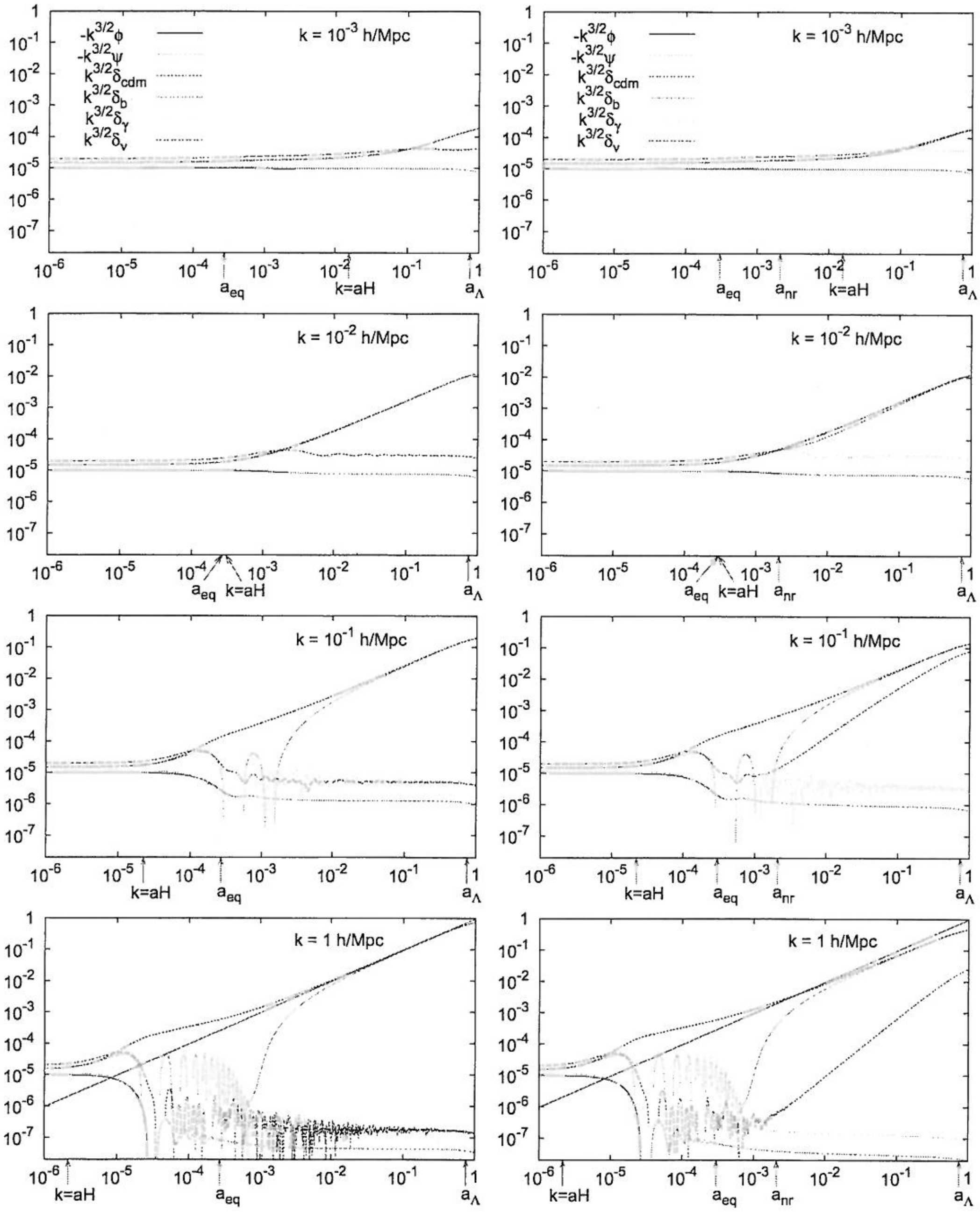


Fig. 11. Evolution of the metric and density perturbations as a function of the scale factor (normalized to $a_0 = 1$ today), in the longitudinal gauge, for modes $10^{-3}\,h\,\mathrm{Mpc}^{-1} < k < 1\,h\,\mathrm{Mpc}^{-1}$ (from top to bottom), and for two cosmological models: ΛCDM (left) and ΛMDM (right), both with $\omega_{\mathrm{m}} = 0.147$ and $\Omega_\Lambda = 0.7$. The integration has been performed with the code CMBFAST starting from the initial condition $k^{3/2}\phi = -10^{-5}$. The ΛMDM model has three degenerate neutrinos with $m_\nu = 0.46$ eV, corresponding to $f_\nu = 0.1$.