In any field, find the strangest thing and then explore it.
(John Archibald Wheeler)

Quantum Fluctuations During Inflation

$$\boxed{v_k = \frac{e^{-ik\tau}}{\sqrt{2}}\left(1 - \frac{i}{k\tau}\right)}$$

Contents

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1 Getting Started

Find an answer to the two questions:

- Why are there small temperature fluctuations ΔT in the cosmic microwave background (CMB), i.e., $\frac{\Delta T}{T} \sim 10^{-5}$, although two points on the last scattering surface with an angular separation
$2\Theta_* \geq 1.19°$ are causally disconnected? (horizon problem)

- How can these small inhomogeneities in CMB be explained?

$\rightarrow$ Inflation:
The shrinking Hubble sphere during inflation, i.e., $\frac{d}{dt}(aH)^{-1} < 1$, solves the horizon problem, see Julian Heeck, *Introduction to Inflation*.
In the following, the smallness of inhomogeneities is explained by quantum fluctuations.
The proceeding starts with an introduction into cosmological perturbation theory with an emphasis on scalar perturbations. After that, the perturbed action $S_{(2)}$ for the inflaton field ϕ minimally coupled to gravity is demonstrated at second-order in the gauge-invariant comoving curvature perturbation $\mathcal{R}$ using comoving gaue. Variation of $S_{(2)}$ yields the Mukhanov-Sasaki equation for the mode functions v_k. Quantizing v_k leads to the unique Bunch-Davies mode functions. The main result of this talk is presented in section 3.5 where the quantum zero-point fluctuations of the Bunch-Davies mode functions in the Minkowski vacuum are discussed from which the power spectrum $P_{\mathcal{R}}(k)$ of the comoving curvature perturbation $\mathcal{R}$ follow. It is $P_{\mathcal{R}}(k)$ which explains the temperature fluctuations in CMB.
The notation of [1] is adopted, in particular, derivatives with respect to physical time t are denoted by overdots, while derivatives with repsect to conformal time τ are indicated by primes.

2 Cosmological Perturbation Theory

This section is dedicated to perturbations generating the observed inhomogeneities. After basic remarks about perturbations the metric and matter perturbations are introduced [2]. Taken the Friedmann-Robertson-Walker (FRW) metric (see Alexander Dueck, *Introduction to Cosmology*) as metric of the background spacetime $\mathcal{N}$ it is stated that general metric and matter perturbations can be decomposed into independent scalar (S), vector (V) and tensor (T) components. Scalar perturbations are examined in more detail yielding the comoving curvature perturbation $\mathcal{R}$. Its power spectrum $P_{\mathcal{R}}(k)$ is the key quantity to describe inhomogeneities arising from inflation.

2.1 General Remarks

- metric of homogeneous, isotropic universe:
$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t)\, d\mathbf{x}^2$
If one allows small perturbations to the metric, then an inhomogeneous universe results.

- Let $\mathcal{M}$ be the physical, i.e., perturbed spacetime with coordinates x^a and let $\mathcal{N}$ be the background, i.e., homogeneous, unperturbed spacetime with coordinates x_b^a. Let $\mathcal{D}$ be the diffeomorphism $\mathcal{D} : \mathcal{N} \to \mathcal{M}$; $x_b^a \mapsto x^a$. Consider a function $X(t, \mathbf{x})$ on $\mathcal{M}$. For a given diffeomorphism $\mathcal{D}$ the perturbation $\delta X(t, \mathbf{x})$ of $X(t, \mathbf{x})$ is
  $\delta X(t, \mathbf{x}) = X(t, \mathbf{x}) - \overline{X}(\mathcal{D}^{-1}(t, \mathbf{x}))$, where $\overline{X}(t)$ indicates a function on the background spacetime $\mathcal{N}$ [3].
  Observe that $\overline{X}(t)$ only depends on time due to homogeneity of $\mathcal{N}$.

- Since the action of general relativity is diffeomorphism invariant, the spacetime $\mathcal{M}$ can also be described in coordinates $\tilde{x}^a$ by the diffeomporphism $\tilde{\mathcal{D}} : \mathcal{N} \to \mathcal{M}$; $x_b^a \mapsto \tilde{x}^a$. $\tilde{\mathcal{D}}$ induces the perturbation
  $\delta \tilde{X}(t, \mathbf{x}) = \tilde{X}(t, \mathbf{x}) - \overline{X}(\tilde{\mathcal{D}}^{-1}(t, \mathbf{x}))$.
  Problem: $\delta \tilde{X}(t, \mathbf{x}) \neq \delta X(t, \mathbf{x})$, however, the action is invariant under diffeomorphism

- gauge transformation $\mathcal{G} : \delta X \xrightarrow{\mathcal{G}} \delta \tilde{X}$
  problem: freedom to chose $\mathcal{G} \to$ arbitrariness in value of perturbation of X at any given spacetime point, unless X is gauge invariant
  $\to$ fix $\mathcal{G}$ by gauge choice

- Different Fourier modes $\delta X(t, \mathbf{k})$ of a perturbation $\delta X(t, \mathbf{x})$, i.e.,
  $\delta X(t, \mathbf{k}) = \int d^3\mathbf{x}\, \delta X(t, \mathbf{x}) e^{i\mathbf{kx}}$, are independent [1].

2.2 Metric and Matter Perturbations

- metric $g_{\mu\nu}$ perturbations:
  $ds^2 = -(1 + 2\Phi)dt^2 + 2a(t)B_i dx^i dt + a^2(t)[(1 - 2\Psi)\delta_{ij} + 2E_{ij}]dx^i dx^j$
  Φ: lapse (3-scalar) $\to \delta g_{00}$
  B_i: shift (3-vector) $\to \delta g_{0i}$
  Ψ: spatial curvature perturbation (3-scalar) $\to \delta g_{ii}$
  E_{ij}: shear (symmetric traceless 3-tensor) $\to \delta g_{ij}$

- matter $T^\mu{}_\nu$ perturbations:
  Energy-momentum tensor $T^\mu{}_\nu$ consists of density ρ with perturbation
  $\delta\rho(t, x^i) = \rho(t, x^i) - \overline{\rho}(t)$,
  pressure p with perturbation
  $\delta p(t, x^i) = p(t, x^i) - \overline{p}(t)$,
  four-velocity u^μ with
  $g_{\mu\nu} u^\mu u^\nu = -1 \to u^\mu = (1 - \Phi, v^i + B^i)$ (perturbed metric used)
  and anisotropic stress $\Sigma^\mu{}_\nu$ with
  $\Sigma^{\mu\nu} u_\nu = 0$.
  The perturbed energy-momentum tensor has the following entries:

$$
\begin{aligned}
T^0{}_0 &= -(\overline{\rho} + \delta\rho)\,, \\
T^0{}_i &= (\overline{\rho} + \overline{p})v_i\,, \\
T^i{}_0 &= -(\overline{\rho} + \overline{p})(v^i + B^i)\,, \\
T^i{}_j &= \delta^i{}_j(\overline{p} + \delta p) + \Sigma^i{}_j\,.
\end{aligned}
$$

  The inflaton ϕ, which is a scalar field (see Julian Heeck, *Introduction to Inflation*), is also pertubed:
  $\delta\phi(t, x^i) = \phi(t, x^i) - \overline{\phi}(t)$.

- The metric perturbations enter the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ and thus the Einstein field equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ link metric and matter perturbations.

2.3 SVT Decomposition

In Fourier space, a perturbation $\delta X(t, \mathbf{k})$ of helicity m has its amplitude multiplied by $e^{im\omega}$ under a rotation with an angle ω around the wavevector $\mathbf{k}$. Scalar, vector and tensor perturbations are defined as perturbations having helicities 0, ± 1, $\pm$ 2, respectively.
The FRW metric is taken as background spacetime $\mathcal{N}$. The symmetries resulting from homogeneity and isotropy allow to decompose the metric and matter perturbations into independent scalar, vector and tensor perturbations. It is a fact that vector perturbations are not created by inflation, hence the consideration is limited to scalar and tensor perturbations.
A 3-scalar α is a helicity scalar, i.e., $\alpha = \alpha^S$. A 3-vector β_i can be decomposed into a helicity scalar β_i^S and a helicity vector β_i^V, i.e., $\beta_i = \beta_i^S + \beta_i^V$ with $\beta_i^S = -\frac{ik_i}{k}\beta$ and $k_i\beta_i^V = 0$ in Fourier space. A symmetric traceless 3-tensor E_{ij} can be decomposed into a helicity scalar E_{ij}^S, a helicity vector E_{ij}^V and a helicity tensor E_{ij}^T i.e., $E_{ij} = E_{ij}^S + E_{ij}^V + E_{ij}^T$ with $E_{ij}^S = \left(-\frac{k_ik_j}{k^2} + \frac{1}{3}\delta_{ij}\right) E$ and $k_iE_{ij}^T = 0$.
Scalar perturbations are discussed in more detail.

2.4 Scalar Perturbations

- four scalar metric perturbations:

$$ds^2 = -(1+2\mathbf{\Phi})dt^2 + 2a(t)\mathbf{B}_{,\mathbf{i}}\,dx^i dt + a^2(t)[(1-2\mathbf{\Psi})\delta_{ij} + 2\mathbf{E}_{,\mathbf{ij}}\,]dx^i dx^j \tag{2.2}$$

  The invariance of ds^2 under a scalar gauge transformation $\mathcal{G}_S$,

$$\begin{aligned} t &\overset{\mathcal{G}_S}{\to} t + \alpha\,, \\ x^i &\overset{\mathcal{G}_S}{\to} x^i + \delta^{ij}\beta_{,j} \end{aligned}$$

  where α and $\beta_{,j}$ are free gauge parameters, directs the transformation behavior of the four scalar metric perturbations:

$$\begin{aligned} \Phi &\overset{\mathcal{G}_S}{\to} \Phi - \dot{\alpha}\,, \\ B &\overset{\mathcal{G}_S}{\to} B + a^{-1}\alpha - a\dot{\beta}\,, \\ E &\overset{\mathcal{G}_S}{\to} E - \beta\,, \\ \Psi &\overset{\mathcal{G}_S}{\to} \Psi + H\alpha\,. \end{aligned}$$

- gauge dependence of matter perturbations:

$$\begin{aligned} \delta\phi &\overset{\mathcal{G}_S}{\to} \delta\phi - \dot{\overline{\phi}}\alpha\,, \\ \delta\rho &\overset{\mathcal{G}_S}{\to} \delta\rho - \dot{\overline{\rho}}\alpha\,, \\ \delta p &\overset{\mathcal{G}_S}{\to} \delta p - \dot{\overline{p}}\alpha\,, \\ \delta q &\overset{\mathcal{G}_S}{\to} \delta q + (\overline{\rho} + \overline{p})\alpha\,, \end{aligned}$$

where q is the scalar part of the 3-momentum density.

- Perturbing the action of the inflaton field ϕ minimally coupled to gravity (see section 3.1),

$$S = \frac{1}{2} \int d^4x \sqrt{g} \left[R - (\nabla\phi)^2 - 2V(\phi) \right], \tag{2.6}$$

  the four scalar metric perturbations and the inflaton perturbation sum up to five scalar degrees of freedom (dof). It is a fact that gauge invariance of (2.6) removes two dof. Constraints from the perturbed Einstein field equations remove two additional dof. One physical degree of freedom remains [4] which is chosen to be the comoving curvature perturbation $\mathcal{R}$,

$$\mathcal{R} = \Psi - \frac{H}{\bar{\rho} + \bar{p}} \delta q \,. \tag{2.7}$$

  From the gauge transformations of Ψ and δq it is clear that $\mathcal{R}$ is gauge invariant.

2.5 Comoving Curvature Perturbation $\mathcal{R}$ and Gaussian Statistics

The time evolution of the comoving curvature perturbation $\mathcal{R}(t, \mathbf{x})$ is governed by the Einstein field equations and energy-momentum conservation $T^{\mu\nu}{}_{;\nu} = 0$:

$$\dot{\mathcal{R}} = -\frac{H}{\bar{\rho} + \bar{p}} \delta p + \frac{k^2}{(aH)^2} \left(... \right) \,.$$

For adiabtic matter perturbations, i.e., $\delta p_{en} = \delta p - \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}} \delta\rho = 0$ which are available for single-field inflation models (see Julian Heeck, *Introduction to Inflation*), $\mathcal{R}$ is conserved on superhorizon scales $k \ll aH$. Calculating the primordial spectrum of $\mathcal{R}$ at horizon crossing, $\mathcal{R}$ can therefore be regarded as time-independent, i.e., $\mathcal{R} = \mathcal{R}(\mathbf{x})$.
In the following $\mathcal{R}(\mathbf{x})$ is assumed to be a Gaussian random field with Fourier transformations

$$\mathcal{R}_{\mathbf{k}} = A \int d^3\mathbf{x} \, \mathcal{R}(\mathbf{x}) e^{-i\mathbf{kx}} \,, \tag{2.8a}$$

$$\mathcal{R}(\mathbf{x}) = B \int d^3\mathbf{k} \, \mathcal{R}_{\mathbf{k}} e^{i\mathbf{kx}} \,, \tag{2.8b}$$

where $BA = \frac{1}{(2\pi)^3}$.

- two-point correlation function $\xi_{\mathcal{R}}(r)$: $\xi_{\mathcal{R}}(r) = \langle \mathcal{R}(\mathbf{x}) \mathcal{R}(\mathbf{x} + \mathbf{r}) \rangle$, $\langle ... \rangle$ means averaging.

- power spectrum $P_{\mathcal{R}}(k)$:

$$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') P_{\mathcal{R}}(k) \tag{2.9}$$

- variance $\sigma^2_{\mathcal{R}}$: $\sigma^2_{\mathcal{R}} = \int d \ln k \, \Delta^2_{\mathcal{R}}(k)$ with $\Delta^2_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2} P_{\mathcal{R}}(k)$ and $B = \frac{1}{(2\pi)^3}$.

In Gaussian statistics, three-point correlation functions vanish and all higher-order correlation functions can be expressed in terms of the two-point correlation function. The power spectrum $P_{\mathcal{R}}(k)$ thus encodes the full information about the fluctuations in the comoving curvature perturbation modes $\mathcal{R}_k$.
In the next section, the modes $\mathcal{R}_k$ are quantized. The fluctuations then naturally emerge from quantum zero-point fluctuations of the quantized mode functions $\hat{v}_{\mathbf{k}}$.

3 Quantum Zero-Point Fluctuations

The aim of this section is to demonstrate how the power spectrum $P_{\mathcal{R}}(k)$ arises from quantum zero-point fluctuations of quantized mode functions $\hat{v}_{\mathbf{k}}$.
The input is the second-order action $S_{(2)}$ for the comoving curvature perturbation $\mathcal{R}$. It is shown that each mode function v_k satisfies the equation of a simple harmonic oscillator (SHO) with time-dependent frequency.
Concerning the time-dependence of the frequencies additional conditions have to be imposed to arrive at the unique Bunch-Davies mode functions v_k.
Since demonstration is the main purpose, full calculations are passed on but the interested reader may find them in the given references.

3.1 Mukhanov-Sasaki Equation

- action (2.6) of inflaton-field matter minimally coupled to gravity:

$$S = \frac{1}{2} \int d^4x \sqrt{g} \left[R - (\nabla\phi)^2 - 2V(\phi)\right]$$

- fixing the gauge parameters α and $\beta_{,j}$ of the scalar gauge transformation $\mathcal{G}_S$:
$\rightarrow$ comoving gauge:

$$\delta\phi = 0,\ g_{ij} = a^2[(1 - 2\mathcal{R})\delta_{ij} + h_{ij}],\ h_{ij,i} = h^i{}_i = 0 \tag{3.1}$$

- scalar perturbations: expanding (2.6) to second order in $\mathcal{R}$ [4] (attention: h_{ij} generates tensor perturbations with resulting gravitational waves [1]):

$$S_{(2)} = \frac{1}{2} \int d^4x \sqrt{g}\, a^3 \frac{\dot{\phi}^2}{H^2} \left[\dot{\mathcal{R}}^2 - a^{-2} \left(\partial_i \mathcal{R}\right)^2\right] \tag{3.2}$$

- variable redefinition

$$v = z\mathcal{R} \tag{3.3}$$

with $z^2 = a^2 \frac{\dot{\phi}^2}{H^2} = 2a^2\epsilon$ and switching to conformal time τ, i.e., $\partial_\tau = \frac{1}{a}\partial_t$ [3]:

$$\Rightarrow S_{(2)} = \frac{1}{2} \int d\tau d^3x \left[(v')^2 - (\partial_i v)^2 + \frac{z''}{z} v^2\right] \tag{3.4}$$

- Expressing (3.4) in Fourier modes $v_{\mathbf{k}}$, i.e.,
$v(\tau, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\, v_{\mathbf{k}}(\tau) e^{i\mathbf{k}\mathbf{x}}$,

and variation after v', $\partial_i v$ and v finally yields the equation for the mode functions v_k ($k = |\mathbf{k}|$), the Mukhanov-Sasaki equation:

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0\,. \tag{3.5}$$

Observe that (3.5) is the equation of a SHO with time-dependent frequency $f(\tau) = k^2 - \frac{z''}{z}$.

3.2 Quantization of the Mode Functions $v_{\mathbf{k}}$

The mode functions $v_{\mathbf{k}}$ are promoted to operators $\hat{v}_{\mathbf{k}}$,

$$\hat{v}_{\mathbf{k}} = v_k(\tau)\hat{a}_{\mathbf{k}} + v^*_{-k}(\tau)\hat{a}^\dagger_{-\mathbf{k}}\,, \tag{3.6}$$

with creation and annihilation operators $\hat{a}^\dagger_{-\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$, respectively, satisfying

$$\hat{a}_{\mathbf{k}} = \frac{W[v_k^*,\, \hat{v}_{\mathbf{k}}]}{W[v_k^*,\, v_k]} \tag{3.7}$$

and

$$[\hat{a}_{\mathbf{k}},\, \hat{a}^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta^{(3)}(\mathbf{k} - \mathbf{k}') \Leftrightarrow W[v_k,\, v_k] = 1\,, \tag{3.8}$$

where $W[v, w] = \frac{i}{\hbar}\left(v^* w' - v^{*\prime} w\right)$.

3.3 Non-Uniqueness of Vacuum State

- Mode functions $v_k(\tau)$ and $v_k^*(\tau)$ in the operator description (3.6) are linear independent solutions of the Mukhanov-Sasaki equation (3.5).
  $\rightarrow$ linear combination of $v_k(\tau)$ and $v_k^*(\tau)$, $\chi_k(\tau) = \alpha_k v_k(\tau) + \beta_k v_k^*(\tau)$, is solution of (3.5).
  Note that for a SHO with time-independent frequenncy the coefficients in the operator description are fixed c numbers, hence creation and annihilation operators are unique.

- If the operator $\hat{v}_{\mathbf{k}}$ is constructed with operators $\hat{a}_{\mathbf{k}}$ and mode functions $v_k(\tau)$, then using a different set of mode functions, e.g., $\chi_k(\tau)$, $\hat{v}_{\mathbf{k}}$ has to be constructed with operators $\hat{b}_{\mathbf{k}}$ according to (3.7):

  $$\hat{v}_{\mathbf{k}} = \chi_k(\tau)\hat{b}_{\mathbf{k}} + \chi^*_{-k}(\tau)\hat{b}^\dagger_{-\mathbf{k}}\,.$$

  $\hat{a}_{\mathbf{k}}$ and $\hat{b}_{\mathbf{k}}$ are related by the Bogolubov transformations [1].

- $\rightarrow$ non-uniqueness of the vaccum state:
  The b-vacuum state, defined by $\hat{b}_{\mathbf{k}}|0\rangle_b = 0$, contains particles created from the a-vacuum state $\hat{a}^\dagger_{\mathbf{k}}|0\rangle_a$: ${}_b\langle 0|\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}|0\rangle_b = |\beta_k|^2\delta(0)$ [5].

- $\rightarrow$ How to calculate zero-point fluctuations $\langle 0|\hat{v}_{\mathbf{k}}\hat{v}_{\mathbf{k}'}|0\rangle$ if mode functions $v_k(\tau)$ and hence the vacuum state $|0\rangle$ are not unique?

3.4 Bunch-Davies Mode Functions v_k

- vacuum state for the fluctuations of $v_k(\tau)$:
  The vacuum state is chosen to be the Minkowski vacuum state $\hat{a}_{\mathbf{k}}|0\rangle = 0$ observed for $\tau \to -\infty$.

- boundary conditions:
  For $z^2 = 2a^2\epsilon$ the following equation holds

$$\frac{z''}{z} = (aH)^2 \left[2 - \epsilon + \frac{3}{2}\eta - \frac{1}{2}\epsilon\eta + \frac{1}{4}\eta^2\eta\kappa\right] \tag{3.9}$$

with $\epsilon = -\frac{\dot{H}}{H^2}$, $\eta = \frac{\dot{\epsilon}}{H\epsilon}$, $\kappa = \frac{\dot{\eta}}{H\eta}$ [6].
In the de Sitter limit, i.e., $\epsilon \to 0$, (3.9) simplifies to

$$\frac{z''}{z} = 2(aH)^2 = \frac{2}{\tau^2}$$

with $a(\tau) = -\frac{1}{H\tau}$.
In the chosen subhorizon limit $\tau \to -\infty$ (3.5) reads

$$v_k'' + k^2 v_k = 0\,,$$

which has oscillating solutions $v_k = \frac{e^{\pm ik\tau}}{\sqrt{2k}}$ The vacuum state $|0\rangle$ is the state with minimum energy for the solution $v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}$.
$\to$ initial condition for all modes:

$$\lim_{\tau \to -\infty} v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}\,. \tag{3.10}$$

In the de Sitter limit (3.5) reads

$$v_k'' + \left(k^2 - \frac{2}{\tau^2}\right) v_k = 0\,,$$

which has the general solution

$$v_k = \alpha \frac{e^{-ik\tau}}{\sqrt{2}} \left(1 - \frac{i}{k\tau}\right) + \beta \frac{e^{ik\tau}}{\sqrt{2}} \left(1 + \frac{i}{k\tau}\right)\,.$$

Observe that α and β are free parameters owing to to the non-uniqueness of the mode functions. However, the subhorizon limit (3.10) sets $\beta = 0$ and the normalization condition (3.8) sets $\alpha = 1$. The unique Bunch-Davies mode functions result:

$$\boxed{v_k = \frac{e^{-ik\tau}}{\sqrt{2}} \left(1 - \frac{i}{k\tau}\right)}$$

with superhorizon limit

$$\lim_{k\tau \to 0} v_k = \frac{1}{i\sqrt{2}} \frac{1}{k^{\frac{3}{2}}\tau}\,. \tag{3.11}$$

3.5 Power Spectrum $P_{\mathcal{R}}(k)$ for Scalar Perturbations from Quantum Fluctuations

- power spectrum $P_v(k)$:
  Using (3.6) and (3.8), the following calculation is obvious:

$$\begin{aligned}
\langle \hat{v}_{\mathbf{k}},\, \hat{v}_{\mathbf{k}'} \rangle &= \langle 0|\hat{v}_{\mathbf{k}},\, \hat{v}_{\mathbf{k}'}|0\rangle \\
&= \langle 0|(v_k(\tau)\hat{a}_{\mathbf{k}} + v^*_{-k}(\tau)\hat{a}^\dagger_{-\mathbf{k}})(v'_k(\tau)\hat{a}_{\mathbf{k}'} + v^*_{-k'}(\tau)\hat{a}^\dagger_{-\mathbf{k}'})|0\rangle \\
&= |v_k|^2\langle 0|[\hat{a}_{\mathbf{k}},\, \hat{a}^\dagger_{-\mathbf{k}'}]|0\rangle \\
&= |v_k|^2\delta(\mathbf{k}+\mathbf{k}') \\
&\equiv P_v(k)\delta(\mathbf{k}+\mathbf{k}')\,.
\end{aligned}$$

  The quantum zero-point fluctuations $\langle \hat{v}_{\mathbf{k}},\, \hat{v}_{\mathbf{k}'} \rangle$ are created on subhorizon scales and freeze on superhorizon scales because the comoving curvature perturbation $\mathcal{R}$ is constant on superhorizon scales (see Figure 1). Since the power spectrum $P_{\mathcal{R}}(k)$ is calculated at horizon crossing the superhorizon limit (3.11) for the mode functions is used yielding
  $P_v(k) = \frac{1}{2k^3}\frac{1}{\tau^2} = \frac{1}{2k^3}(aH)^2$.

- power spectrum $P_{\mathcal{R}}(k)$:
  Using $v = z\mathcal{R}$, $P_{\mathcal{R}}(k)$ equals

$$\begin{aligned}
P_{\mathcal{R}} &= \frac{1}{z^2}P_v \\
&= \frac{1}{2a^2\epsilon_\star}\frac{1}{2k^3}(a_\star H_\star)^2 \\
&= \frac{1}{4k^3}\frac{H_\star^2}{\epsilon_\star} \\
\Rightarrow P_{\mathcal{R}}(k) &= \frac{1}{2k^3}\frac{H_\star^4}{\dot{\phi}_\star^2}
\end{aligned}$$

  with the relation $\epsilon = \frac{1}{2}\frac{\dot{\phi}^2}{H^2}$ for a scalar field ϕ with action (2.6). Quantities with lower index $\star$ are evaluated at the time of horizon crossing.

- Combining equations (3.6), (3.3), (2.8a), (2.8b), (3.1) and (2.7), the outcome is that the scalar metric perturbations (2.2) invoked to explain inhomogeneities are explained by quantum zero-point fluctuations.

4 $P_{\mathcal{R}}(k)$ and the Cosmic Microwave Background

The power spectrum $P_{\mathcal{R}}(k)$ is related to the observed angualr power spectrum C_l of CMB temperature fluctuations:

$$C_l = \frac{2}{\pi}\int dk\, k^2\, P_{\mathcal{R}}(k)\Delta_{Tl}(k)^2$$

with transfer function $\Delta_{Tl}(k)$, see Michael Duerr, *Inflation and Contact with Observations.*

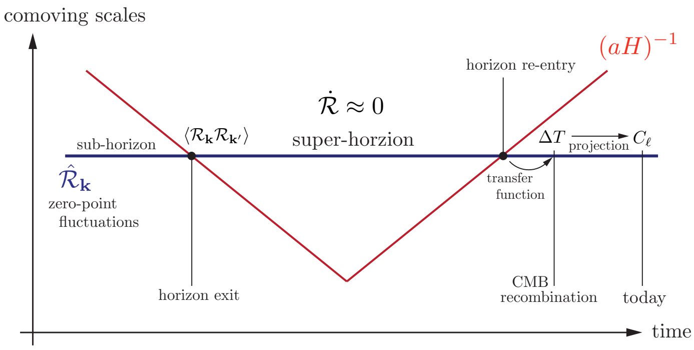

Figure 1: Creation and evolution of perturbations in the inflationary universe. Fluctuations are created quantum mechanically on subhorizon scales (see section 3.5). While comoving scales, k^{-1}, remain constant the comoving Hubble radius during inflation, $(aH)^{-1}$, shrinks and the perturbations exit the horizon and freeze until horizon re-entry at late times. After horizon re-entry the fluctuations evolve into anisotropies in the CMB and perturbations in the LSS. This time-evolution has to be accounted for to relate cosmological observations to the primordial perturbations laid down by inflation (see Michael Duerr, *Inflation and Contact with Observations*). Figure from [1].

5 Guide to the Literature

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   - description of gauge-invariant variables and their physical meaning
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   - derivation of the second-order action for the comoving curvature perturbation $\mathcal{R}$
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6. Lecture notes of the inflation course at the Summer School in Cosmology, ICTP Trieste (2010), by Daniel Baumann